Modern Physics Letters A
Vol. 29, No. 34 (2014) 1450183 (11 pages)

DOI: 10.1142/S0217732314501831

On colored HOMFLY polynomials for twist knots

Andrei Mironov[*,†,‡], Alexei Morozov[†,‡,§] and Andrey Morozov[†,‡,¶]

[*]*Lebedev Physics Institute, Moscow 119991, Russia*
[†]*ITEP, Moscow 117218, Russia*
[‡]*National Research Nuclear University MEPhI, Moscow 115409, Russia*
[¶]*Moscow State University, Moscow 119991, Russia*
[¶]*Laboratory of Quantum Topology, Chelyabinsk State University, Chelyabinsk 454001, Russia*
[*]*mironov@itep.ru, mironov@lpi.ru*
[§]*morozov@itep.ru*
[¶]*Andrey.Morozov@itep.ru*

Received 18 August 2014
Accepted 3 September 2014
Published 30 October 2014

Recent results of Gu and Jockers provide the lacking initial conditions for the evolution method in the case of the first nontrivially colored HOMFLY polynomials $H_{[21]}$ for the family of twist knots. We describe this application of the evolution method, which finally allows one to penetrate through the wall between (anti)symmetric and non-rectangular representations for a whole family. We reveal the necessary deformation of the differential expansion, what, together with the recently suggested matrix model approach gives new opportunities to guess what it could be for a generic representation, at least for the family of twist knots.

Keywords: Chern–Simons theory; colored HOMFLY polynomials.

PACS Nos.: 03.65.Ca, 03.65.Fd

1. Introduction

Calculation of colored HOMFLY polynomials remains one of the biggest problems in modern quantum field theory. These are Wilson-loop averages in $3d$ Chern–Simons (CS) theory,[1,2]

$$H_R^{\mathcal{L}} = \left\langle \mathrm{Tr}_R\, P \exp\left(\oint_{\mathcal{L}} \mathcal{A}\right)\right\rangle \tag{1}$$

with "colored" meaning that the gauge field $\mathcal{A}$ in the P-exponent is in an arbitrary representation R of the gauge group $\mathrm{SU}(N)$. Representation dependence of Wilson-loop averages is highly nontrivial and very informative: for example, in confinement

[§]Corresponding author

phase of QCD, the average in the fundamental representation obeys the area law, while in the adjoint one, it is just the perimeter law. Chern–Simons theory is topological, therefore there is no room for metric dependencies (like area or perimeter laws), instead the averages depend in a sophisticated way on topology (linking) of the closed contour $\mathcal{L}$ (which is called knot or link depending on the number of connected components).

Amusingly, besides the direct analogy, there is also a transcendental relation to Yang–Mills theory in higher dimensions: knot polynomials are made from the Racah matrices of quantum groups, which define the modular transformation of $2d$ conformal blocks, and these are related through the celebrated AGT relations[3–12] to S-dualities between various $4d$ and $5d$ supersymmetric Yang–Mills models.

A dream-like solution to the colored HOMFLY problem is known for the special class of torus knots $\mathcal{L} = [m, n]$: they are provided by the action of the simplest cut-and-join operator

$$\hat{W}_{[2]} = \frac{1}{2}\sum_{a,b=1}^{\infty}\left((a+b)p_a p_b \frac{\partial}{\partial p_{a+b}} + ab p_{a+b}\frac{\partial^2}{\partial p_a \partial p_b}\right) \tag{2}$$

on the Adams-transformed characters (Schur functions) $\chi_R\{p_k\}$:

$$H_R^{[m,n]} = q^{\frac{n}{m}\hat{W}_{[2]}}\chi_R\{p_{mk}\}\Big|_{p_k=p_k^*=\frac{\{A^k\}}{\{q^k\}}}\,. \tag{3}$$

Here $q = \exp\left(\frac{2\pi i}{g+N}\right)$ is made from the gauge coupling g, parameter $A = q^N$, and we use the standard notation $\{x\} = x - x^{-1}$, so that the quantum number is $[x] = \frac{\{q^x\}}{\{q\}}$. A remarkable fact about the Wilson averages in CS theory in the simply connected spacetime R^3 or S^3 is that they are *polynomials* in q and A, see Ref. 13 for the latest attempt to explain this remarkable property. The Adams transformation is the relabeling of time-variables $p_k \to p_{mk}$.

This Rosso–Jones formula (3), which we presented in the version of Ref. 15, is valid in this form for the torus *knots*, i.e. when m and n are co-prime. For the l-component links (l is the biggest common divisor of m and n) the single character is substituted by the product of l different characters (which can all be in different representations). This formula allows several important reformulations: in terms of TBEM eigenvalue matrix model,[16,17] in terms of evolution along the "time" n (Refs. 15 and 18) and in terms of differential expansion of Refs. 19 and 20. It also allows a straightforward deformation to superpolynomials,[15,21,22] and, along the lines of Refs. 23–25, perhaps, also to Khovanov–Rozansky polynomials (where the answers are known from an alternative approach of Refs. 26 and 27).

Of course, Eq. (3) is extremely inspiring: one can think about writing something similar for other knots, by making use of a similar continuation from the topological locus p_k^* to arbitrary times p_k (Ref. 28) and extending $\hat{W}_{[2]}$ evolution to that, generated by other cut-and-join operators from Refs. 29 and 30. This Hurwitz-τ-function[31–35] description of knot polynomials looks promising, but only the first attempts were made.[36–38] The main obstacle is the lack of explicit examples. An available calculation tool is essentially the old Reshetikhin–Turaev approach,[39–44] either in its more traditional form of skein relations and Racah calculus, developed very far in Refs. 45–54, or in the modernized version,[55–58] based on use of the universal $\mathcal{R}$-matrices and often supplemented by the old cabling method in the new version of Ref. 59. The real problem, however, is that in these tedious calculations, one does not immediately see the new structures, which are so obvious in Ref. 14, and which are the real target of the knot/CS theory studies.

A first real breakthrough after the old result (3) was the discovery[60] of the general formula, all totally symmetric $R = [r]$ and antisymmetric $R = [1^r]$ representations for the figure-eight knot 4_1, which was immediately generalized to entire one-parametric family of twist knots:[18]

$$H_{[r]}^{(k)} = 1 + \sum_{s=1}^{r} \frac{[r]!}{[s]![r-s]!} F_s^{(k)}(A|q) \prod_{j=0}^{s-1} \underbrace{\{Aq^{r+j}\}\{Aq^{j-1}\}}_{Z_{r|1}^{(j)}} . \tag{4}$$

For 4_1 ($k = -1$) all the coefficient functions are unities, $F_s^{(-1)} = 1$, for the trefoil 3_1 ($k = 1$), which is the only torus knot among the twist ones, $F_s^{(1)} = (-A^2)^s q^{s(s-1)}$ while for generic integer k

$$F_s^{(k)} = q^{s(s-1)/2} A^s \sum_{j=0}^{s} (-)^j \frac{[s]!}{[j]![s-j]!} \frac{\{Aq^{2j-1}\}(Aq^{j-1})^{2jk}}{\prod_{i=j-1}^{s+j-1} \{Aq^i\}} . \tag{5}$$

In fact, the family of twist knots can be further extended,[18] see also Refs. 61 and 62. A very clear structure of differential expansion is seen in (4) — in fact much more transparent than in the torus case,[20] which allowed one to make further conjecture about arbitrary rectangular representations $R = [s^r]$.[20,63] Also straightforward was generalization to superpolynomials,[18,60] what allowed one to check a conjecture about their representation dependence[22,64] and to develop the theory of (super)A-polynomials,[61,62,65–67] generalizing the old story originally known in the Jones case.[68–73] The latest achievement comes from the study of (4) which is an inspiring attempt to generalize the TBEM matrix model from the torus to twist knots.[74] If it was fully successful, it would provide *all* colored HOMFLY polynomials for twist knots (with arbitrary Young diagram R), in its present form, it allows one only to check the first terms of the $\hbar$-expansion ($q = e^\hbar$), what, in fact, is not so bad.

The problem is that going beyond the (anti)symmetric and especially beyond rectangular representations is extremely hard: the problem for particular knots and even particular N is at the border of capacity of available computers already for $R = [21]$, nothing to say about the most interesting case of $R = [31]$. Actually, since[59] $H_{[21]}$ for the four different twisted knots were known, this was not enough to apply the evolution method of Ref. 18. This is now possible, because in Ref. 53 $H_{[21]}$ were calculated for three more twist knots, so we can finally look for general formulas. Also a partial validation is provided by the recent matrix model of Ref. 74. It is the purpose of this paper to describe the result: it is provided by Eqs. (9) and

(10) and still needs to be converted in the differential expansion form, generalizing Eq. (21) to arbitrary k. Even in this unfinished form, this is a next big step after the guesswork of Ref. 75, it provides a support to the ideas of that paper and puts the study of colored knot polynomials on a more solid ground. Also, representation [21] is the first non-rectangular one, and it is the first representation distinguishing, for example, the mutant pair of the Kinoshita–Terasaka and Conway knots.[76–79]

2. HOMFLY for $R = [21]$ by Evaluation Method

In application to twist knots, the evolution method explained in full detail in Ref. 18 consists of three steps.

(a) Decompose the product of representation $R = [21]$ and its conjugate $\bar{R} = \overline{[21]} = [2^{N-2}1]$ into irreducible ones:

$$[21] \otimes \overline{[21]} = [432^{N-4}1] \oplus [42^{N-2}] \oplus [42^{N-3}11] \oplus [332^{N-3}] \oplus [332^{N-4}11] \oplus 2 \times [32^{N-2}1] \oplus [2^N]\,. \tag{6}$$

Note that $[32^{N-2}1]$ comes with nontrivial multiplicity (2), what never happens for (anti)symmetric representations, but is the generic case in the study of colored HOMFLY polynomials.

One can check this decomposition by summing up dimensions

$$\begin{aligned}
D_{[432^{N-4}1]}(N) &= \frac{N!\frac{(N+1)!}{2\cdot 3}(N+1)(N+2)(N+3)}{3\cdot(N-4)!(N-3+1)(N-2+2)\cdot\frac{(N-3+1)!}{2}(N-2+2)(N-1+3)} \\
&= \frac{(N^2-9)(N^2-1)^2}{9}\,, \\
D_{[42^{N-2}]}(N) &= \frac{N!\frac{(N+1)!}{2}(N+2)(N+3)}{2\cdot(N-2+1)!(N-1+3)(N-2)!(N-1+3)} = \frac{N^2(N-1)(N+3)}{4}\,, \\
D_{[42^{N-3}11]}(N) &= D_{[31^{N-3}]}(N) = \frac{\frac{N!}{2}(N+1)(N+2)}{2(N-3)!(N-2+2)} = \frac{(N^2-4)(N^2-1)}{4}\,, \\
D_{[332^{N-3}]} &= \frac{N!\frac{(N+1)!}{2}(N+2)(N+1)}{2(N-3)!(N-2+1)(N-1+1)(N-3+1)!(N-2+2)(N-1+2)} \\
&= \frac{(N^2-4)(N^2-1)}{4}\,, \\
D_{[332^{N-4}11]} &= D_{[221^{N-4}]} = \frac{\frac{N!}{2}(N+1)N}{2(N-4)!(N-3+1)(N-2+1)} = \frac{(N+1)N^2(N-3)}{4}\,, \\
D_{[32^{N-2}1]}(N) &= D_{[21^{N-2}]} = \frac{N!(N+1)}{(N-2)!(N-1+1)} = N^2-1\,, \\
D_{[2^N]} &= D_{[0]} = 1
\end{aligned} \tag{7}$$

the sum is indeed equal to the square of $D_{[21]}(N) = \frac{N(N^2-1)}{3}$.

(b) The k-dependence of $H^{(k)}$ is dictated be the eigenvalues of $\hat{W}_{[2]}$, $\varkappa_R = \sum_{(i,j)\in R}(i-j) - \sum_{(i,j)\in[2^N]}(i-j)$:

$$\begin{aligned}
\varkappa_{[432^{N-4}1]} &= 3+2+1+(N-1)+(N-2)+(N-3) = 3N \Rightarrow A^3\,,\\
\varkappa_{[42^{N-2}]} &= 3+2+(N-1)+(N-2) = 2N+2 \Rightarrow q^2A^2\,,\\
\varkappa_{[42^{N-3}11]} &= 3+2+(N-2)+(N-3) = 2N \Rightarrow A^2\,,\\
\varkappa_{[332^{N-3}]} &= 2+1+(N-1)+(N-2) = 2N \Rightarrow A^2\,,\\
\varkappa_{[332^{N-4}11]} &= 2+1+(N-2)+(N-3) = 2N-2 \Rightarrow q^{-2}A^2\,,\\
\varkappa_{[32^{N-2}1]} &= 2+(N-2) = N \Rightarrow A\,,\\
\varkappa_{[2^N]} &= 0 \Rightarrow 1\,.
\end{aligned} \tag{8}$$

This is because they are actually the eigenvalues of quantum $\mathcal{R}$-matrix, which acts as the unit operator in each irreducible representation in the 2-strand channel.

In other words, one makes the following anzatz for the evolution along the k-variable:

$$H^{(k)}_{[21]} = u_3A^{6k} + (u_{2p}q^{4k} + u_{20} + u_{2m}q^{-4k})A^{4k} + u_1A^{2k} + u_0\,. \tag{9}$$

(c) The six unknown k-independent coefficients $u_\alpha(A,q)$ in (9) can be now defined from "the initial conditions": the actual values of $H^{(k)}_{[21]}$ for particular values of k. To determine the six parameters, one needs six explicitly known answers for $H^{(k)}_{[21]}$. Immediately available are two: for the unknot at $k=0$ and for the torus knot, trefoil at $k=1$. Two more, for the 3-strand knots 4_1 at $k=-1$ and 5_2 at $k=2$ were found by a tedious cabling calculation in Ref. 59, but this was not enough. Quite recently, Jie Gu and Hans Jockers published their results[53] for the 4-strand 6_1 ($k=-2$) and 7_2 ($k=3$) and even the 5-strand 8_1 ($k=-3$): they use an alternative group theoretical approach *a la*,[45–52] and are not restricted with the number of strands in the closed braid since these use plat representations of knots. This allows us not only to apply the evolution method, but even provides the seventh point in the k-line to check the outcome.

The answer is:

$$u_3 = -A^3\frac{\{Aq^3\}\{A/q^3\}\{Aq\}\{A/q\}}{\{A\}}\,,$$

$$u_{2p} = \frac{[3]}{[2]^2}A^3\frac{\{Aq^3\}\{A\}}{\{Aq\}}([2]A^2q^{-3} - (q^4+1-q^{-2}+q^{-4}))\,,$$

$$u_2 = -2\{q\}^2A^3\left(\frac{[3]}{[2]}\right)^2\frac{\{Aq^2\}\{A/q^2\}}{\{A\}}\,,$$

$$u_{2m} = \frac{[3]}{[2]^2}A^3\frac{\{Aq^{-3}\}\{A\}}{\{Aq^{-1}\}}([2]A^2q^3 - (q^4+1-q^2+q^{-4}))\,,$$

$$u_1 = -\frac{[3]A^3}{\{A\}}(A^4 - (q^6+q^{-6})A^2 + (2q^6-4q^4+4q^2-3+4q^{-2}-4q^{-4}+2q^{-6}))\,,$$

$$u_0 = \frac{A^3}{\{Aq\}\{A\}\{A/q\}}\left(A^6 - \frac{[3][10]}{[2][5]}A^4 + \frac{[3][10]^2}{[2]^2[5]^2}A^2 - \frac{[10][14]}{[2]^2[5][7]}\right). \tag{10}$$

Now, when we possess the general expression for $H^{(k)}_{[21]}$, it is easy to check some of its crucial properties enumerated in Ref. 75.

3. Checking Elementary Properties

For $A = q^2$ the gauge group is SL(2), for which there is no difference between representations [21] and [1], thus (9) and (10) coincide with

$$H^{(k)}_{[1]} = 1 + F^{(k)}_1\{Aq\}\{A/q\} = 1 - \frac{A^{k+1}\{A^k\}}{\{A\}}\{Aq\}\{A/q\}$$

$$\xrightarrow{A=q^2} \; J^{(k)}_{[1]}(q) = \frac{q^2+q^6+(1-q^6)\cdot q^{4k}}{1+q^2}\,. \tag{11}$$

The same coincidence takes place at $A = q^{-2}$, this follows also from the symmetry

$$H^{(k)}_{[21]}(A,q^{-1}) = H^{(k)}_{[21]}(A,q)\,. \tag{12}$$

For $A = q$ one could expect that the knot polynomial vanishes, and this is indeed true, but for the *unreduced* HOMFLY polynomial. As to (9) and (10), it is the *reduced* polynomial, and nothing special happens to it at $A = q$: what vanishes at $A = q$ is the quantum dimension of representation [21], i.e. $\chi^*_{[21]} = \frac{\{Aq\}\{A\}\{A/q\}}{\{\{q\}^2\{q^3\}}$.

At $A = 1$, one obtains the Alexander polynomial

$$H^{(k)}_{[21]}(A=1,q) = 1 + k\{q^3\}^2 \;=\; H^{(k)}_{[1]}(A=1,q^3) \tag{13}$$

(the last relation[15,60,80] holds only for *hook* diagrams, but representation [21] is of that type).

Finally, for $q = 1$, one gets the *special* polynomial, and

$$H^{(k)}_{[21]}(q=1,A) = (A^2\cdot(1-A^{2k-1}\{A\}))^3 = (H^{(k)}_{[1]}(q=1,A))^3 \tag{14}$$

in full accordance with Refs. 15, 64 and 80.

4. Checking Consistency with the Matrix Model

Another comparison to make is with the matrix model suggestion of Ref. 74. Since there are no reason for any doubt about the answer (9) and (10) for $H^{(k)}_{[21]}$, this is rather a check of the matrix model. However, an advantage of the matrix model approach is that its calculation complexity is almost independent of the representation, and, once developed, it provides generic colored polynomials. Thus, checks and insights about this approach are extremely important for practical calculations, not only for pure theory.

The claim of Ref. 74 is that the colored Jones polynomial (i.e. HOMFLY polynomial at $N = 2$, $A = q^2$) possesses a remarkable integral representation:

$$J_r(q = e^{\hbar}) = H_{[r-1]}(A = q^2, q = e^{\hbar}) \sim \int e^{-\frac{u^2}{2\gamma\hbar}} \sinh(ru) \underbrace{e^{-\frac{\gamma\hbar}{2}\partial_u^2} \mathcal{J}\left(\frac{u}{\gamma}\middle|\hbar\right)}_{\nu(u)}, \tag{15}$$

where $\mathcal{J}(\rho|\hbar) = J_{r=\rho/\hbar}(q = e^{\hbar})$ is the same Jones polynomial, only with variables changed to describe the vicinity of the large-representation (Kashaev) limit. Explicit expressions for $\mathcal{J}$ are difficult to get, even if *some* formulas (like hypergeometric series of Refs. 68–71) are known for generic colored Jones polynomials, but differential expansion like (4) is exactly what is needed for this purpose,

An immediate lift of (15) to arbitrary N,

$$H_R(q = q^{\hbar}, A = e^{\hbar N}) \overset{?}{\sim} \int \chi_R[e^u] \prod_{i<j}^{N} (\nu(u_i - u_j) \cdot \sinh(u_i - u_j)) \prod_{i=1}^{N} e^{-\frac{u_i^2}{\gamma\hbar}}\, du_i \tag{16}$$

is known[16,17] to give the colored HOMFLY polynomials for the torus knots, but cannot do so for generic knots, because the HOMFLY polynomial cannot be reconstructed from the Jones one in a knot-independent way. Already for the twisted knots, there are corrections[74] to (16), but they start from the order $\hbar^5$ and seem to have a controllable dependence on R and k, which is currently under investigation. At present, one can use this technique to find the $\hbar$ expansion of H_R up to the terms $\hbar^6$ and for $N \leq 5$.

We performed this check and made sure that (9) and (10) are in this sense consistent with Ref. 74, what is not at all trivial, because the input in Ref. 74 is only from knowledge of the HOMFLY polynomials in symmetric representations. In particular, we confirmed, that the first nonvanishing correction to (16) is given by the factor (38) of Ref. 74 with r substituted by $|R|$ (the number of boxes in R, i.e. 3 for $R = [21]$):

$$\begin{gathered} 1 + 2(N-2)(3N-4)(2N\varkappa_R + |R|(N^2 - |R|))U^{(k)}\hbar^5 + O(\hbar^6)\,, \\ U^{(k)} = k(k+1)(4k-1) + \frac{48k^2}{\gamma^3}(39k^2 - 13k + 1)\,. \end{gathered} \tag{17}$$

The parameter γ (which is equal to $-mn$ in the matrix model of Refs. 16 and 17 for the $[m, n]$ torus knot, e.g. $U^{(1)}(\gamma = -6) = 0$ for the trefoil) remains unspecified: all formulas of Ref. 74 hold for arbitrary value of γ, which can still be adjusted, perhaps, with some other free parameters of similar type, to get rid of corrections like (17).

5. On Differential Expansion for $H^{(k)}_{[21]}$

Despite all these successes with Eqs. (9) and (10), they are still far from looking like the basic formula (4), i.e. are not yet represented in the desired form of the differential expansion *a la* (Refs. 18, 20 and 60), which is a q-deformation of the binomial expansion for the *special* polynomial

$$H_R^{(k)}(q=1,A) = (H_\Box^{(k)}(q=1,A))^{|R|} = (1-(A^{2k}-1)(A^2-1))^{|R|}$$
$$= \sum_{j=0}^{|R|} C^j_{|R|}(-(A^{2k}-1)(A^2-1))^j \,. \tag{18}$$

From Ref. 75 we know that the answer should be rewritten in the form

$$H^{(k)} = 1 + (Z_{2|0}+Z_{3|3}+Z_{0|2})\cdot F_1^{(k)}(A) + \{A\}\cdot Z_{2|2}\cdot G^{(k)}(A|q)\,, \tag{19}$$

where

$$F_1^{(k)} = -A^2\frac{A^{2k}-1}{A^2-1} \tag{20}$$

does not depend on q.

Nice representations of this type are known for $k=\pm 1$:

$$\begin{aligned} H^{(-1)}_{[21]} &= 1 + (Z_{2|0}+Z_{3|3}+Z_{0|2}) + Z_{2|2}(Z_{4|0}+Z_{0|4}+Z_{0|0}) \\ &\quad + Z_{3|3}Z_{2|2}Z_{0|0} - \{q\}^2 Z_{2|2}Z_{0|0}\,, \\ H^{(-1)}_{[21]} &= 1\,, \\ H^{(1)}_{[21]} &= 1 - A^2(Z_{2|0}+Z_{3|3}+Z_{0|2}) + A^4 Z_{2|2}(q^3Z_{3|0}+q^{-3}Z_{0|3}+Z_{0|0}) \\ &\quad - A^6 Z_{3|3}Z_{2|2}Z_{0|0} + A^4(1+A^2)\{q\}^2 Z_{2|2}Z_{0|0}\,. \end{aligned} \tag{21}$$

However, what should be the right representation for $G^{(k)}(A|q)$ for generic k remains unclear, this adds to the problems with the choice of the differential expansion for the torus knots reported in Ref. 20. The situation is unclear even at the level of (reduced) Alexander polynomial: at $A=1$, one has

$$G^{(k)}(A=1) = \sum_{i=0}^{2k}[4k+1-2i]\cdot u_i^{(k)}\,, \tag{22}$$

where the coefficients are almost independent of k, but look somewhat ugly.

$$u_i^{(k)} = u_i^{(k-1)} \quad \text{for } i \le 2k-4\,,$$
$$u_{2k-3}^{(k)} = -\frac{1}{6}k(4k^2-21k+23) \quad \text{for } k>1\,,$$

$$u_{2k-2}^{(k)} = 8 + \frac{1}{6}(k-2)(4k^2 - 7k + 45) \text{ for } k > 1\,, \quad u_0^{(1)} = 1\,,$$

$$u_{2k-1}^{(k)} = 2 - \frac{1}{6}(k+1)(4k^2 - 19k + 18)\,,$$

$$u_{2k}^{(k)} = 1 + \frac{1}{6}(k-1)(4k^2 - 5k + 18)\,. \tag{23}$$

We plan to return to discussion of different options here in a separate publication.

Acknowledgments

We are indebted to Jie Gu and Hans Jockers for important checks and additional efforts to enlarge the list of calculated quantities and to Satoshi Nawata for pointing out a misprint in our main formula (10). Our work is partly supported by grant NSh-1500.2014.2, by RFBR grants 13-02-00457 (A.Mir.), 13-02-00478 (Al.Mor.), 14-02-00627 (And.Mor.), 14-01-31395_young_a (And.Mor.), by joint grants 13-02-91371-ST and 14-01-92691-Ind, by the Brazil National Counsel of Scientific and Technological Development (Al.Mor.), by the Laboratory of Quantum Topology of Chelyabinsk State University (Russian Federation government grant 14.Z50.31.0020) (And.Mor.) and by the Dynasty Foundation (And.Mor.).

References

1. S. Chern and J. Simons, *Proc. Nat. Acad. Sci.* **68**, 791794 (1971); *Ann. Math.* **99**, 48 (1974).
2. E. Witten, *Commun. Math. Phys.* **121**, 351 (1989).
3. L. F. Alday, D. Gaiotto and Y. Tachikawa, *Lett. Math. Phys.* **91**, 167 (2010).
4. N. Wyllard, *JHEP* **0911**, 002 (2009).
5. A. Mironov and A. Morozov, *Nucl. Phys. B* **825**, 1 (2009).
6. V. A. Alba, V. A. Fateev, A. V. Litvinov and G. M. Tarnopolsky, *Lett. Math. Phys.* **98**, 33 (2011).
7. A. Mironov, A. Morozov and Sh. Shakirov, *JHEP* **1103**, 102 (2011).
8. A. Mironov, A. Morozov and Sh. Shakirov, *JHEP* **1102**, 067 (2011).
9. A. Belavin and V. Belavin, *Nucl. Phys. B* **850**, 199 (2011).
10. A. Morozov and A. Smirnov, *Lett. Math. Phys.* **104**, 585 (2014).
11. S. Mironov, An. Morozov and Y. Zenkevich, *JETP Lett.* **99**, 115 (2014).
12. Y. Matsuo, C. Rim and H. Zhang, arXiv:1405.3141.
13. E. Witten, arXiv:1401.6996.
14. M. Rosso and V. F. R. Jones, *J. Knot Theory Ramifications* **2**, 97 (1993).
15. P. Dunin-Barkowski, A. Mironov, A. Morozov, A. Sleptsov and A. Smirnov, *JHEP* **03**, 021 (2013).
16. M. Tierz, *Mod. Phys. Lett. A* **19**, 1365 (2004).
17. A. Brini, B. Eynard and M. Mariño, arXiv:1105.2012.
18. A. Mironov, A. Morozov and And. Morozov, *AIP Conf. Proc.* **1562**, 123 (2013).
19. N. M. Dunfield, S. Gukov and J. Rasmussen, *Experimental Math.* **15**, 129 (2006).
20. S. Arthamonov, A. Mironov and A. Morozov, *Theor. Math. Phys.* **179**, 509 (2014).
21. M. Aganagic and Sh. Shakirov, arXiv:1105.5117.
22. I. Cherednik, arXiv:1111.6195.

23. V. Dolotin and A. Morozov, *Nucl. Phys. B* **878**, 12 (2014).
24. A. Anokhina and A. Morozov, *JHEP* **2014**, 63 (2014).
25. A. Morozov, And. Morozov and Ant. Morozov, arXiv:1407.6319.
26. E. Gorsky, A. Oblomkov and J. Rasmussen, arXiv:1206.2226.
27. E. Gorsky and L. Lewark, arXiv:1404.0623.
28. A. Mironov, A. Morozov and And. Morozov, *Strings, Gauge Fields, and the Geometry Behind: The Legacy of Maximilian Kreuzer*, eds. A. Rebhan, L. Katzarkov, J. Knapp, R. Rashkov and E. Scheidegger (World Scientific Publishing, 2013), pp. 101–118.
29. A. Mironov, A. Morozov and S. Natanzon, *Theor. Math. Phys.* **166**, 1 (2011).
30. A. Mironov, A. Morozov and S. Natanzon, *J. Geom. Phys.* **62**, 148 (2012).
31. S. Kharchev, A. Marshakov, A. Mironov and A. Morozov, *Int. J. Mod. Phys. A* **10**, 2015 (1995).
32. A. Orlov and D. M. Shcherbin, *Theor. Math. Phys.* **128**, 906 (2001).
33. A. Orlov, *Theor. Math. Phys.* **146**, 183 (2006).
34. A. Alexandrov, A. Mironov, A. Morozov and S. Natanzon, *J. Phys. A: Math. Theor.* **45**, 045209 (2012).
35. A. Alexandrov, A. Mironov, A. Morozov and S. Natanzon, arXiv:1405.1395.
36. A. Mironov, A. Morozov and A. Sleptsov, *Theor. Math. Phys.* **177**, 1435 (2013).
37. A. Mironov, A. Morozov and A. Sleptsov, *Eur. Phys. J. C* **73**, 2492 (2013).
38. A. Mironov, A. Morozov, A. Sleptsov and A. Smirnov, arXiv:1310.7622.
39. N. Reshetikhin, LOMI preprint, E-4-87, E-17-87 (1987), http://math.berkeley.edu/reshetik/.
40. V. Turaev, *Invent. Math.* **92**, 527 (1988).
41. N. Reshetikhin and V. Turaev, *Commun. Math. Phys.* **127**, 1 (1990).
42. E. Guadagnini, M. Martellini and M. Mintchev, in Clausthal 1989, Proceedings, Quantum groups, pp. 307–317.
43. E. Guadagnini, M. Martellini and M. Mintchev, *Phys. Lett. B* **235**, 275 (1990).
44. A. Morozov and A. Smirnov, *Nucl. Phys. B* **835**, 284 (2010).
45. R. K. Kaul and T. R. Govindarajan, *Nucl. Phys. B* **380**, 293 (1992).
46. P. Ramadevi, T. R. Govindarajan and R. K. Kaul, *Nucl. Phys. B* **402**, 548 (1993).
47. P. Ramadevi, T. R. Govindarajan and R. K. Kaul, *Nucl. Phys. B* **422**, 291 (1994).
48. P. Ramadevi and T. Sarkar, *Nucl. Phys. B* **600**, 487 (2001).
49. Zodinmawia and P. Ramadevi, arXiv:1107.3918; arXiv:1209.1346.
50. S. Nawata, P. Ramadevi and Zodinmawia, arXiv:1302.5143.
51. S. Nawata, P. Ramadevi and Zodinmawia, arXiv:1302.5144.
52. S. Nawata, P. Ramadevi and Zodinmawia, *JHEP* **1401**, 126 (2014).
53. J. Gu and H. Jockers, arXiv:1407.5643v2.
54. Sh. Deguchi, arXiv:1408.2212.
55. A. Mironov, A. Morozov and And. Morozov, *JHEP* **03**, 034 (2012).
56. H. Itoyama, A. Mironov, A. Morozov and And. Morozov, *JHEP* **2012**, 131 (2012).
57. H. Itoyama, A. Mironov, A. Morozov and And. Morozov, *IJMPA* **28**, 1340009 (2013).
58. A. Anokhina, A. Mironov, A. Morozov and And. Morozov, *Adv. High Energy Phys.* **2013**, 931830 (2013).
59. A. Anokhina and And. Morozov, *Teor. Mat. Fiz.* **178**, 3 (2014).
60. H. Itoyama, A. Mironov, A. Morozov and And. Morozov, *JHEP* **2012**, 131 (2012).
61. S. Nawata, P. Ramadevi, Zodinmawia and X. Sun, arXiv:1209.1409.
62. H. Fuji, S. Gukov, M. Stosic and P. Sulkowski, arXiv:1209.1416.
63. E.Gorsky, S. Gukov and M. Stosic, arXiv:1304.3481.
64. Ant. Morozov, *JHEP* **1212**, 116 (2012); *JETP Lett.* **97**, 171 (2013).
65. H. Fuji, S. Gukov and P. Sulkowski, arXiv:1203.2182; arXiv:1205.1515.

66. M. Aganagic, T. Ekholm, L. Ng and C. Vafa, arXiv:1304.5778.
67. S. Arthamonov, A. Mironov, A. Morozov and And. Morozov, *JHEP* **04**, 156 (2014).
68. R. Gelca, *Math. Proc. Cambridge Philos. Soc.* **133**, 311 (2002).
69. R. Gelca and J. Sain, *J. Knot Theory Ramifications* **12**, 187 (2003).
70. S. Gukov, *Commun. Math. Phys.* **255**, 577 (2005).
71. S. Garoufalidis, *Geom. Topol. Monogr.* **7**, 291 (2004).
72. D. Galakhov, A. Mironov, A. Morozov and A. Smirnov, *Theor. Math. Phys.* **172**, 939 (2012).
73. A. Mironov and A. Morozov, *AIP Conf. Proc.* **1483**, 189 (2012).
74. A. Alexandrov, A. Mironov, A. Morozov and And. Morozov, arXiv:1407.3754.
75. A. Anokhina, A. Mironov, A. Morozov and And. Morozov, *Nucl. Phys. B* **882C**, 171 (2014).
76. H. R. Morton and H. J. Ryder, *Geom. Topol. Monogr.* **1**, 365 (1998).
77. H. R. Morton and M. Rampichini, Knots in Hellas 98, *Proc. Int. Conf. on Knot Theory and Its Ramifications*, eds. C. Gordon *et al.* (World Scientific, 2000), pp. 335–346.
78. H. R. Morton and R. J. Hadji, *Algebr. Geom. Topol.* **2**, 11 (2002).
79. H. Morton and S. Lukac, *J. Knot Theory Ramifications* **12**, 395 (2003).
80. Sh. Zhu, arXiv:1206.5886.