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# TABULATING KNOTS IN THE THICKENED KLEIN BOTTLE

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**Abstract:** We tabulate all knots in the oriented thickened Klein bottle having diagrams with three crossings and less. For proving that the knots are distinct, we use a generalization of the Kauffman bracket polynomial in four variables.

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**Keywords:** thickened Klein bottle, knot, generalized Kauffman polynomial

**1. Introduction.** The article is devoted to composing the table of knots in the thickened Klein bottle $K\tilde{\times}I$ whose minimal diagrams have at most 3 crossings. First, we construct regular graphs of degree 4 having at most 3 vertices. For each graph, we enumerate the projections that are homeomorphic to it and then the corresponding knot diagrams. Proving that most of the so-obtained knots are distinct became possible with the use of the constructed useful invariant of knots in the thickened Klein bottle—an analog of the Kauffman polynomial [1]. The obtained table contains 33 knots. So far tables of global knots, i.e., knots in three-dimensional manifolds different from the sphere, were constructed only for the projective space [2], the thickened torus [3], and the solid torus [4]. This article is a first attempt to study and tabulate knots in the thickened Klein bottle.

**2. Projections and diagrams of knots in $K\tilde{\times}I$.** The thickened Klein bottle is the orientable skew product $K\tilde{\times}I$ of the Klein bottle $K$ and the segment $I$. A *knot* in $K\tilde{\times}I$ is an arbitrary simple closed curve in $\mathrm{Int}(K\tilde{\times}I)$. Two knots $k_1, k_2 \subset K\tilde{\times}I$ are called *equivalent* if there exists a homeomorphism of $K\tilde{\times}I$ onto itself taking $k_1$ to $k_2$. Like classical knots, it is convenient to define knots in $K\tilde{\times}I$ by projections and marked projections, i.e. diagrams.

Definition 1. A connected graph $G \subset K$ is called a *projection* if its every vertex has degree 4 and the straight-ahead rule determines a cyclic walk composed of all edges of $G$.

Of course, the projections of classical knots possess this properties. Therefore, this definition of the projections of a knot in $K\tilde{\times}I$ is fully justified. Note that the conventional way of defining the classical knots on using projections with discontinuities near the vertices, i.e. diagrams, does not work in our case. The reason is that the Klein bottle $K$ is nonorientable. Therefore, the notions “above-below,” needed for marking the crossings, are undefined.

For obviating this obstacle, represent $K$ as a square whose lateral edges are identified by parallel translation and the bases are identified by the composition of a parallel translation and the symmetry with respect to the middle of the base. Moreover, all vertices of the projection $G$ under consideration of the given knot $k$ must get inside the square. Then $G$ is a collection of proper arcs in the square such that, under the indicated identifications, the ends of the arcs are glued pairwise. The diagram of a knot differs from its projection by the circumstance that the vertices in the projection are marked in the above-mentioned conventional way. Since the square is orientable, the notions “above-below” are well-defined. Examples projections and diagrams of the knot in $K\tilde{\times}I$ are presented in Fig. 1.

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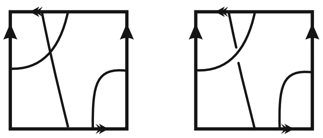

Fig. 1

It is well known that, in composing tables of the classical knots, one tries to confine oneself to the prime knots since, by Schubert's theorem, all other knots are uniquely representable as connected sums of prime knots. For small tables, the primarity of the table knots is checked but it is not so for the knots with at least 16 crossings. We will also try to decrease the number of the projections under consideration and the corresponding knots in $K\tilde{\times}I$ by discarding the projections and knots that can be obtained from those already constructed by easy operations.

Definition 2. A diagram of a knot $k \subset K\tilde{\times}I$ is called *minimal* if its complexity (number of crossings) does not exceed the complexity of any diagram of any knot equivalent to $k$. A projection $G \subset K$ is called *minimal* if at least one of the corresponding diagrams is minimal.

Definition 3. A projection $G \subset K$ is called *reducible* if the complement of $G$ contains a nontrivial simple closed curve.

Definition 4. A projection $G \subset K$ is called *composite* if there exists a disk $D \subset K$ such that the intersection $G \cap \partial D$ consists of two points and $D$ contains a vertex of $G$.

Definition 5. A minimal projection $G \subset K$ is called *essential* if $G$ is neither composite nor reducible.

In tabulating the projections, the reader can confine oneself to essential projections. Indeed, by Definition 2, all diagrams corresponding to a nonminimal projection define knots having projections with less vertices and, therefore, have already occurred before. It is better to tabulate the reducible projections separately since each of them is contained either in an annulus or in a Möbius strip lying in $K$. Thus, we can compose the tables of projections in the annulus and the Möbius strip and take into account all nonequivalent embeddings of these surfaces into $K$. Composite projections can be neglected because the knots corresponding to them result from knots of less complexity by taking connected sums with the classical knots.

### 3. The graphic types of projections in $K$.

**Lemma 1** [3]. *There exist exactly* 6 *types of regular graphs of degree* 4 *having at most* 3 *vertices and* 2 *loops* (*Fig.* 2).

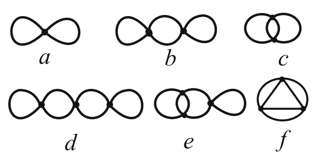


Fig. 2

### 4. The main results.

**Theorem 1.** *There are exactly* 17 *different essential projections in $K$ whose number of vertices is at most* 3. *They are all depicted in Fig.* 3.

**Theorem 2.** *There exist at most* 33 *different essential knots in $K\tilde{\times}I$ having diagrams with at most 3 crossings* (*Fig.* 4).

We will need the operation of vertex removal which is depicted in Fig. 5. Orient the edges of the projection by using any of its cyclic walks in accordance with the "straight-ahead" rule. Then the operation of the removal of a given vertex consists in replacing this vertex by a dotted arc as shown in the figure. In result, we obtain a new graph, which has one vertex less and is endowed with a dotted arc.

We will need one more vertex removal operation. Suppose that a projection $G$ includes two embedded circles which intersect exactly at one vertex and do it untransversally. Then the removal operation consists in a small shift of circles so that this vertex be stretched to a dotted arc joining these circles (in Fig. 6, one circle is decorated and the other is not).

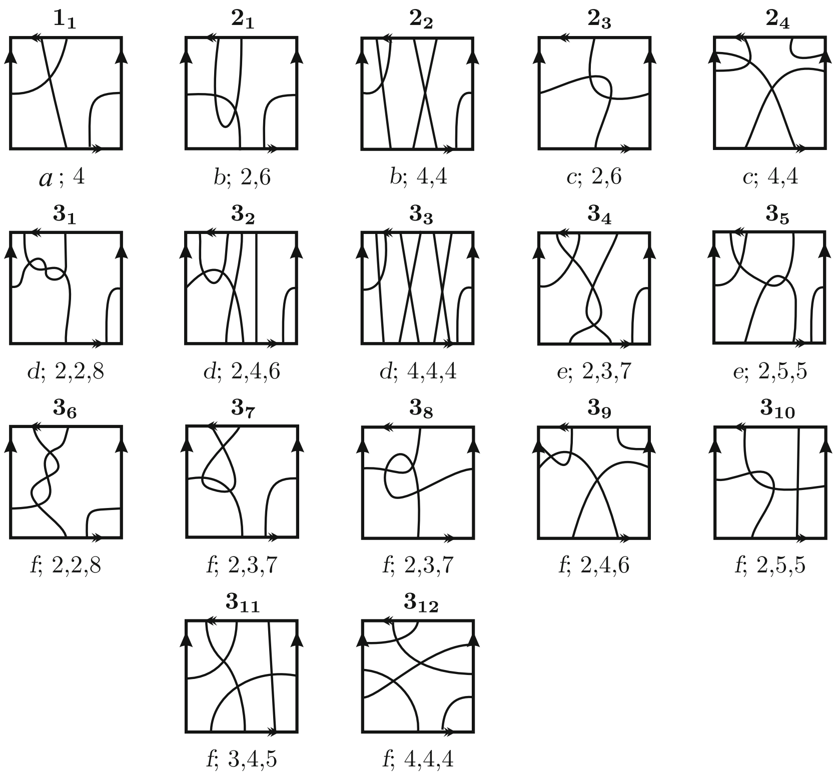


Fig. 3. Projections in $K$ with at most 3 vertices.
Over each projection, its number is indicated and under it,
its graphic type (see Fig. 2) and the face-vector composed of
the numbers of the angles of its faces are given

Proof of Theorem 1. Decompose the proof into the six cases in accordance with the number of graphic types of projections with at most 3 vertices (see Fig. 2):

**1**. Consider an essential projection $G$ of type $(a)$. Since its vertex is separating, the removal of the vertex gives a pair of circles in $K$ joined by a dotted arc. Since an arc has only two endpoints, both circles are one-sided. Otherwise, one of them would have a free side, along which we could draw a nontrivial closed simple curve in $K \setminus G$. This contradicts the essentiality of the projection. The circles can be joined by only one dotted line (up to homeomorphisms $(K, G) \to (K, G)$ (see the connection scheme in Fig. 7 on the left). Contracting the dotted line to a point, we obtain projection $1_1$ (see Fig. 3). For the analogous reason, there are no essential projections with 3 loops.

**2**. Consider an essential projection $G$ of type $(b)$. Removing its both vertices, we obtain a chain of three circles joined by two dotted arcs. As above, the two extreme circles are one-sided, and the remaining circle can be either trivial or not. This is shown on the middle and rightmost schemes in Fig 7. Contracting the arc to points, we obtain projections $2_1$ and $2_2$ (see Fig. 3).

**3**. Consider an essential projection $G$ of type $(c)$. Removing one of its vertices, we obtain a pair of circles that intersect transversally at one point and are joined by dotted arc. Up to homeomorphism, $K$ contains only two such pairs: $(m, \mu)$ and $(m_1, m_2)$, where $\mu$ is a meridian of the bottle $K$ while $m$, $m_1$, and $m_2$ are one-sided circles. In both case, the dotted arcs are drawn in a unique manner (Fig. 8). Contracting them to points, we obtain projections $2_3$ and $2_4$ (see Fig. 3).

**4**. Constructing essential projections of type $(d)$ is similar to constructing projection of types $(a)$ and $(b)$. Removing its all three vertices gives a chain of four circles joined by three dotted arcs. As above, the extreme circles must be one-sided. The middle circles can be as trivial as nontrivial. Thus, we have

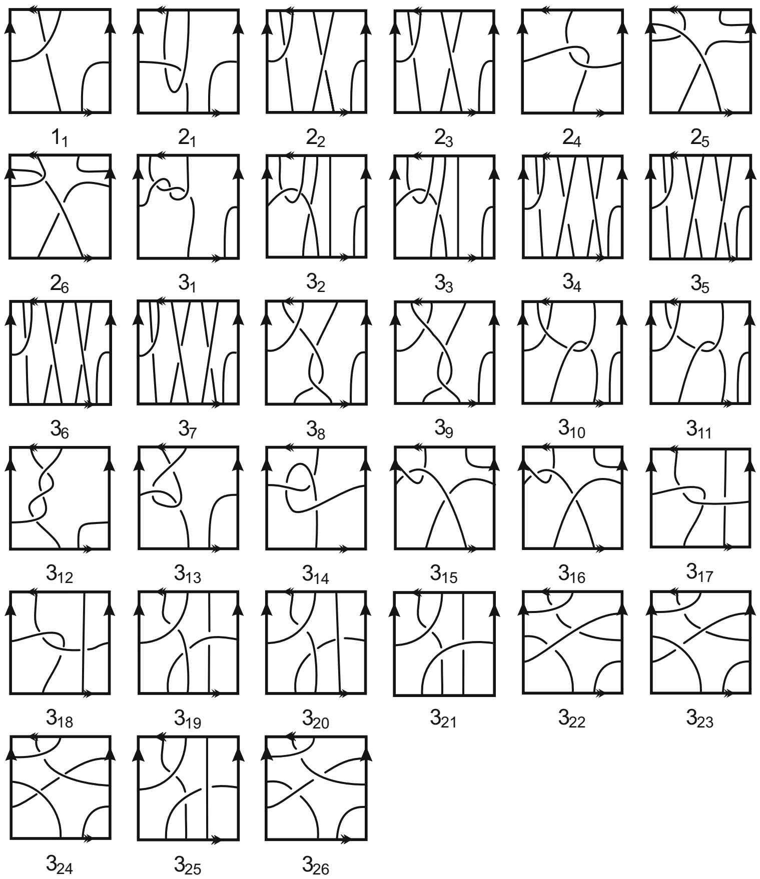


Fig. 4. Essential knots in $K\tilde{\times}I$ with at most 3 crossings

three possibilities for the disposition of circles, which it is convenient to encode as follows: $(m, t, t, m)$, $(m, t, \lambda, m)$, and $(m, \lambda, \lambda, m)$, where $m$ stands for one-sided circles, $t$ denotes trivial circles, and $\lambda$ denotes longitudes in $K$. The corresponding schemes are shown in Fig. 9. The contraction of the dotted arcs to points gives the three new projections $3_1$, $3_2$, and $3_3$ (see Fig. 3).

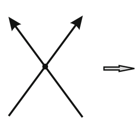

Fig. 5. Removal of a vertex

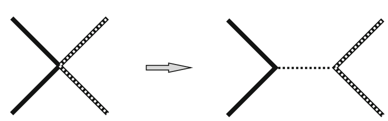

Fig. 6. Removal of a nontransversal intersection point of two circles

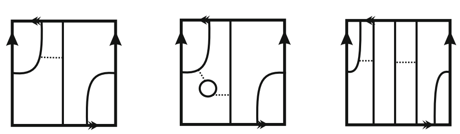

Fig. 7. Schemes of projections of types $(a)$ and $(b)$

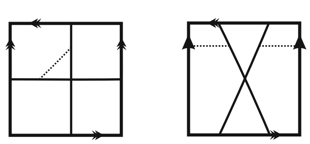

Fig. 8. Schemes of projections of type $(c)$

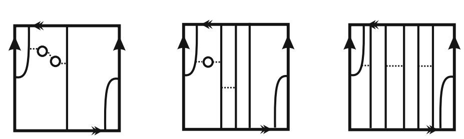

Fig. 9. Schemes of projections of type $(d)$

**5**. Let $G$ be a projection of type $(e)$. Removing the vertex of its loop and one of the two remaining vertices, we obtain the three circles $c_1$, $c_2$, and $c_3$. Two of them, say $c_1$ and $c_2$, intersect transversally at one point and are joined by a dotted arc. The circle $c_3$ is located separately and joined by a dotted arc with one of the first two circles. As was observed above, two circles in $K$ can intersect transversally at one point only in the two cases:

1. One of the circles is a meridian; i.e. it has type $\mu$; and the other is one-sided.
2. Both circles are one-sided.

In the first case, $c_3$ must be trivial, but then $G$ will have a trivial loop, which contradicts the essentiality of $G$. In the second case, $c_3$ must also reverse orientation because it cannot be trivial or have type $\lambda$ since $G$ is an essential projection. Note that the complement in $K$ to the union of $c_1$, $c_2$, and $c_3$ consists of an open disk and an open annulus. Therefore, it is not hard to prove that the dotted arcs joining the circles can be drawn only in two nonequivalent ways (Fig. 10). Contraction of the dotted arcs gives projections $3_4$ and $3_5$.

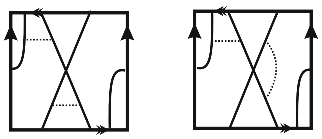

Fig. 10. Schemes of projections of type $(e)$

**6**. Let $G$ be a projection of type $(f)$. It is a chain of three circles $c_1, c_2, c_3 \subset G$ every two of which have one common vertex. Assume that, at some vertex, the circles passing through it (let them be $c_1$ and $c_2$) intersect transversally. But then, at none of the remaining two vertices, the intersection of the circles can be transversal since otherwise we would have the projection of a link and not of a knot.

As above, the two cases are possible:

1. One of the circles $c_1$ and $c_2$ is a meridian, i.e. it has type $\mu$; and the other reverses orientation, i.e. it has type $m$.
2. Both circles $c_1$ and $c_2$ have type $m$.

Removing both nontransversal points of intersection of $c_3$ with $c_1$ and $c_2$, we obtain a new circle $\tilde{c}_3$ (a slightly displaced copy of $c_3$), which is joined by disjoint dotted arcs with $c_1$ and $c_2$. In the first case, $\tilde{c}_3$ can be only trivial, and in the second, it can be either trivial or one-sided. The schemes of the corresponding three projections are shown in Fig. 11 (the first 3 schemes). Contracting the dotted arcs to points, we obtain projection $3_8$ in the first case and projections $3_9$ and $3_{11}$ in the second case.

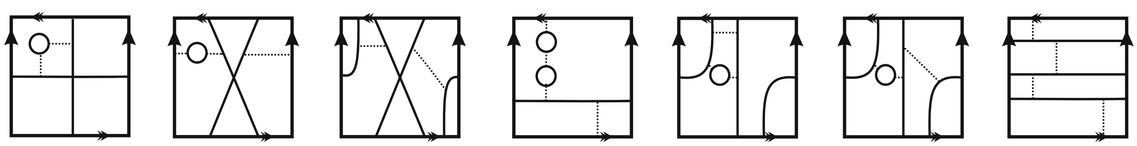

Fig. 11. Schemes of projections of type $(f)$

Suppose that all three intersection points of the circles $c_1, c_2, c_3 \subset G$ are not transversal. Removing them, we arrive at a closed chain of three disjoint circles joined by three dotted arcs. It is convenient to encode the potentially possible cases by triples of types of involved circles.

1. $(\mu, t, t)$. One circle is a meridian but the other two are trivial.
2. $(m, m, t)$. Two circles are one-sided but the third is trivial.
3. $(\mu, \mu, \mu)$. All three circles are meridians.

The schemes of the corresponding four projections are shown in Fig. 11 (the last 4 schemes). In contracting the dotted arcs, in the first case, we obtain projection $3_6$, and in the second, projections $3_7$ and $3_{11}$ (whose schemes differ by the disposition of dotted arcs), in the third case—projection $3_{12}$.

Since the longitude $\lambda$ decomposes $K$ into two Möbius strips, the triples containing $\lambda$ cannot lead to essential projections. The remaining triples $(\mu, t, t)$, $(m, t, t)$ and $(t, t, t)$ give projections of links. Therefore, there are no other projections of type $(f)$.

For proving Theorem 2, we will need a new invariant of knots in the thickened Klein bottle, which can be called the *generalized Kauffman polynomial.* In contrast to the conventional normalized Kauffman bracket of one variable (see [1]), we use four variables $A$, $x$, $y$, and $z$ which are needed for taking into account the number and types of circles in $K$ obtained after resolving all crossings in the diagram in accordance with the chosen state $s$. The exact formula is as follows:

$$X(K) = (-A)^{-3w(K)} \sum_s A^{\alpha(s)-\beta(s)}(-A^2 - A^{-2})^{\gamma(s)} x^{\delta_1(s)} y^{\delta_2(s)} z^{\delta_3(s)},$$

where $\alpha(s)$ and $\beta(s)$ are the numbers of the markers $A$ and $B$ in $s$, $\gamma(s)$ is the number of trivial circles on the Klein bottle obtained by resolving all crossings, and $\delta_1(s)$, $\delta_2(s)$, and $\delta_3(s)$ are the numbers of the circles of types $\mu, \lambda$, and $m$ respectively. The sum is taken over all possible states, $w(K)$ stands for the twisting number of the diagram. The proof of the fact that this polynomial is an invariant of the knot coincides practically with the proof of the invariance of the Kauffman polynomial for classical knots given in [1] and hence is omitted.

Proof of Theorem 2. In order to construct diagrams and knots corresponding to a given projection, we must mark all its crossings by discontinuities. Since, in changing all crossing by the opposite crossings, the equivalence class of the knot remains the same, each projection with three vertices can give at most four different knots. Moreover, if the projection has a biangular face then the number of the corresponding knots is diminished to two, and if this biangular face bounds a triangular face by an edge then only one knot remains. In result, we obtain 33 diagrams.

Below, we give the table of the Kauffman polynomials for all knots defined by diagrams in Fig. 4. Here it is worth noting that if the diagrams of two knots differ from each other by the change of type for all crossings then their Kauffman polynomials differ by the change of signs of all powers of $A$. We assume these polynomials identical.

$1_1$: $-A^4x - A^{-2}z^2$,
$2_1$: $A^{-8}x + (A^{-6} - A^{-2})z^2$,
$2_2$: $A^{-8}x + 2A^{-6}z^2 + A^{-4}yz^2$ ,
$2_3$: $x + (A^{-2} + A^2)z^2 + yz^2$,
$2_4$: $(A^4 + A^6 - A^{10})z$,
$2_5$: $(-2A^4 - 2A^8) + A^4y + A^8x^2$,
$2_6$: $(-A^{-4} - 2 - A^4) + y + x^2$,
$3_1$: $-A^{-12}x + (-A^{-10} + A^{-6} - A^{-2})z^2$,
$3_2$: $-A^{-4}x + (-A^{-6} - A^{-2} + A^2)z^2 + (-A^{-4} + 1)yz^2$,
$3_3$: $-A^{-12}x + (-2A^{-10} + A^{-6})z^2 + (A^{-8} - A^{-4})yz^2$,
$3_4$: $-A^{-4}x + (-A^{-6} - A^{-2} + A^2)z^2 - 2A^{-4}yz^2 - A^{-2}y^2z^2$,
$3_5$: $-A^{-12}x + (-2A^{-10} + A^{-6})z^2 - 2A^{-8}yz^2 - A^{-6}y^2z^2$,
$3_6$: $-A^4x - A^2z^2 + (-1 - A^4)yz^2 - A^2y^2z^2$,
$3_7$: $-A^{-4}x - A^{-2}z^2 + (-A^{-4} - 1)yz^2 - A^{-2}y^2z^2$,
$3_8$: $(-1 + A^6 - A^{10})z - A^2yz$,
$3_9$: $(A^6 - A^8 - A^{10})z - A^6yz$,
$3_{10}$: $(A^4 - A^{10} + A^{14})z + (-A^8 + A^{12})yz$,
$3_{11}$: $(-A^2 + A^4 + A^6)z + (-A^4 + A^8)yz$,
$3_{12}$: $(-A^{-16} + A^{-12} - A^{-8})x - A^{-6}z^2$,
$3_{13}$: $(A^4 - A^8 + A^{12})x + (-A^{10} + A^{14})z^2$,
$3_{14}$: $(A^{-2} + 1 - A^2 - A^4 + A^8)z$,
$3_{15}$: $(-A^{-2} + A^2 + 2A^6) + (A^{-2} - A^2)y - A^6x^2$,
$3_{16}$: $(-A^2 + 2A^{10} + A^{14}) + (A^6 - A^{10})y - A^{10}x^2$,
$3_{17}$: $(-2A^8 + A^{12})x + (-A^6 - A^{10} + A^{14})z^2$,
$3_{18}$: $(-1 - A^4 + A^8)x + (-2A^2 + A^6)z^2$,
$3_{19}$: $(A^{-8} - 2A^{-4} - A^{-2} + A^2)z - A^{-2}yz$,
$3_{20}$: $A^4z - A^2yz$,

$3_{21}$: $(-A^{-10} - A^{-8} + A^{-6})z - A^{-6}yz$,
$3_{22}$: $3A^{-12}x - A^{-6}z^2 - A^{-12}x^3$,
$3_{23}$: $(A^{-8} + A^{-4} + 1)x - A^{-2}z^2 - A^{-4}x^3$,
$3_{24}$: $(1 + A^4 + A^8)x - A^2z^2 - A^4x^3$,
$3_{25}$: $A^{-4}z - A^{-2}yz$,
$3_{26}$: $(A^{-8} + A^{-4} + 1)x - A^{-2}z^2 - A^{-4}x^3$.

REMARK. It is not hard to prove that knots $3_{20}$ and $3_{25}$ are equivalent. For this, it suffices to carry one of the crossings in the diagram of $3_{25}$ over through the upper base and turn the diagram by 180 degrees. The same applies to knots $3_{23}$ and $3_{24}$.

The authors have failed to prove that knots $3_6$ and $3_7$, which have identical Kauffman polynomials. The same applies to knots $3_{23}$ and $3_{26}$.

From the table of Kauffman polynomials we see that all remaining knots have different polynomials. Therefore, there are no duplicates among them.

**Conjecture.** *Knot $3_6$ is different from knot $3_7$, and knot $3_{24}$ is different from knot $3_{26}$.*

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