International Journal of Modern Physics A
Vol. 31, No. 27 (2016) 1650156 (21 pages)

DOI: 10.1142/S0217751X16501566

# Vassiliev invariants for pretzel knots

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Received 1 September 2016
Revised 19 September 2016
Accepted 20 September 2016
Published 30 September 2016

We compute Vassiliev invariants up to order six for arbitrary pretzel knots, which depend on $g+1$ parameters $n_1, \ldots, n_{g+1}$. These invariants are symmetric polynomials in $n_1, \ldots, n_{g+1}$ whose degree coincide with their order. We also discuss their topological and integer-valued properties.

*Keywords*: Knot theory; Vassiliev invariants; pretzel knots; HOMFLY polynomials.

PACS numbers: 11.15.Yc, 02.10.Kn

## 1. Introduction

A set of Vassiliev invariants is conjectured to be a complete invariant of a knot as well as a set of colored quantum invariants. Despite these two sets were discovered approximately simultaneously (see Refs. 1–9), a progress in the construction and the calculation of colored quantum invariants is significantly greater than of Vassiliev invariants. About calculations of quantum invariants during last years see, for example, Refs. 10–18 for latest reviews about Vassiliev invariants. One of the most successful approach to quantum invariant calculations is to divide all knots into families and try to find explicit answers for them. It turns out in many cases that it is rather easy to compute or sometimes guess quantum invariants for particular family and the answer turns out to be amazingly simple and well-structured. This phenomenon can be illustrated by a famous family of torus knots which all colored HOMFLY polynomials are given by the beautiful Rosso–Jones formula.[19,20] This formula inspires many mathematicians to begin their research with torus knots. In particular, there were calculated Vassiliev invariants of torus knots up to order 6 in Ref. 21. Recently,[22,23] it obtained some explicit results for quantum invariants of pretzel knots which are a natural generalization of simplest torus knots of a form $T[2, 2n+1]$ to a Riemann surface of arbitrary genus $g$. It stimulates us to compute and discuss their Vassiliev invariants.

## 2. Vassiliev Invariants from Chern–Simons Theory

The incorporation of Vassiliev invariants in the path-integral representation is clear from the following picture. Let $A$ be a connection on $\mathbb{R}^3$ taking values in some representation $R$ of a Lie algebra $g$, i.e. in components:

$$A = A_i^a(x) T^a \, dx^i \,,$$

where $T^a$ are the generators of $g$. Let curve $C$ in $\mathbb{R}^3$ give a particular realization of knot $K$. Consider the holonomy of $A$ along $C$, it is given by the ordered exponent:

$$\Gamma(C, A) = P \exp \oint_C A = 1 + \oint_C A_i^a(x) T^a + \oint_C A_{i_1}^{a_1}(x_1) \int_0^{x_1} A_{i_2}^{a_2}(x_2) T^{a_1} T^{a_2} + \cdots \,.$$

The Wilson loop along $C$ is a function depending on $C$ and $A$ defined as a trace of holonomy:

$$W_R(C, A) = \mathrm{tr}_R \, \Gamma(C, A) \,.$$

According to Ref. 24, there exists a functional $S_{\mathrm{CS}}(A)$ (we write it down explicitly later) such that the integral averaging of the Wilson loop with the weight $\exp\left(-\frac{2\pi i}{\hbar} S(A)\right)$ has the following remarkable property:

$$\boxed{\langle W_R(K) \rangle = \frac{1}{Z} \int DA \exp\left(-\frac{2\pi i}{\hbar} S_{\mathrm{CS}}(A)\right) W_R(C, A)} \,, \tag{1}$$

where

$$Z = \int DA \exp\left(-\frac{2\pi i}{\hbar} S_{\mathrm{CS}}(A)\right) ,$$

i.e. the averaging of $W_R(C, A)$ with the weight $\exp\left(-\frac{2\pi i}{\hbar} S(A)\right)$ does not depend on the realization $C$ of the knot in $\mathbb{R}^3$ but only on the topological class of equivalence of knot $K$ (in what follows we will denote the averaging of quantity $Q$ with this weight by $\langle Q \rangle$) and therefore, $\langle W(K) \rangle$ defines a knot invariant.

The distinguished Chern–Simons action giving the invariant average (1) has the following form:

$$S_{\mathrm{CS}}(A) = \int_{\mathbb{R}^3} \mathrm{tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right) . \tag{2}$$

If we normalize the algebra generators $T^a$ as $\mathrm{tr}(T^a T^b) = \delta^{ab}$ and define the structure constants $f$ of algebra $g$ as $[T^a, T^b] = f_{abc} T^c$, then the action takes the form:

$$S_{\mathrm{CS}}(A) = \epsilon^{ijk} \int_{\mathbb{R}^3} dx^3 \, A_i^a \partial_j A_k^a + \frac{1}{6} f_{abc} A_i^a A_j^b A_k^c \,.$$

Formula (1) is precisely the path integral representation of knot invariants. It is believed that all invariants of knots can be derived from this expression. Let us outline the appearance of Vassiliev invariants in this scheme. Obviously the mean value $\langle W(K)\rangle$ has the following structure:

$$\begin{aligned}\langle W(C,A)\rangle &= \left\langle \sum_{n=0}^{\infty} \oint dx_1 \int dx_2 \cdots \int dx_n \right. \\ &\qquad \left. \times A^{a_1}(x_1)A^{a_2}(x_2)\cdots A^{a_3}(x_n)\ \mathrm{tr}(T^{a_1}T^{a_2}\cdots T^{a_n}) \right\rangle \\ &= \sum_{n=0}^{\infty} \oint dx_1 \int dx_2 \cdots \int dx_n \\ &\qquad \times \langle A^{a_1}(x_1)A^{a_2}(x_2)\cdots A^{a_3}(x_n)\rangle\, \mathrm{tr}\big(T^{a_1}T^{a_2}\cdots T^{a_n}\big) \\ &= \sum_{n=0}^{\infty} \sum_{m=1}^{N_n} V_{n,m} G_{n,m}\,. \end{aligned} \tag{3}$$

From this expansion, we see that the information about the knot and the gauge group enter in $\langle W(K)\rangle$ separately. The information about the embedding of a knot into $\mathbb{R}^3$ is encoded in the integrals of the form:

$$V_{n,m} \sim \oint dx_1 \int dx_2 \cdots \int dx_n \langle A^{a_1}(x_1)A^{a_2}(x_2)\cdots A^{a_3}(x_n)\rangle$$

and the information about the gauge group and representation enter in the answer as the "group factors":

$$G_{n,m} \sim \mathrm{tr}(T^{a_1}T^{a_2}\cdots T^{a_n})\,.$$

$G_{k,m}$ are the group factors called *chord diagrams with $n$ chords*. Chord diagrams with $n$ chords form a vector space $H_n$. Despite $\langle W(K)\rangle$ being a knot invariant, the numbers $V_{n,m}$ are not invariants. This is because the group elements $G_{n,m}$ are not independent, and the coefficients $V_{n,m}$ are invariants only up to relations among $G_{n,m}$. Dimensions of $H_n$ are summarized in the table:

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $\dim(H_n)$ | 1 | 1 | 1 | 3 | 4 | 9 |

(4)

In order to pass to Vassiliev invariants, we have to choose some basis in the space of chord diagrams. We do it following Ref. 25, refer to that paper for details. The so-called trivalent diagrams are introduced in a way represented for orders two and three in Fig. 1. Group-theoretical rules for graphical representation of chords and trivalent diagrams are presented in Fig. 2. For the general definition of trivalent diagrams refer to Ref. 25, see also Ref. 26.

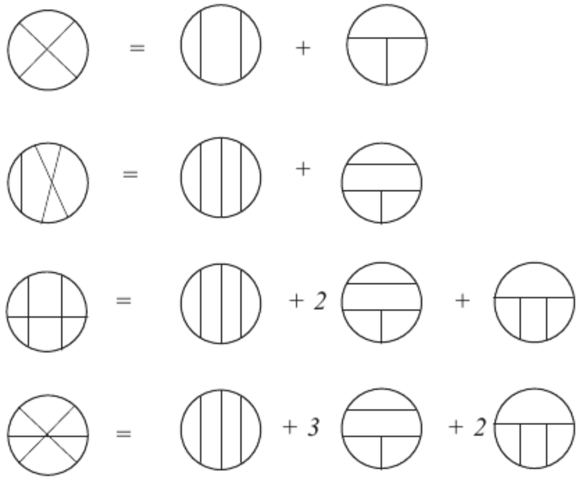


Fig. 1. Relation between trivalent diagrams and chord diagrams up to order 3.

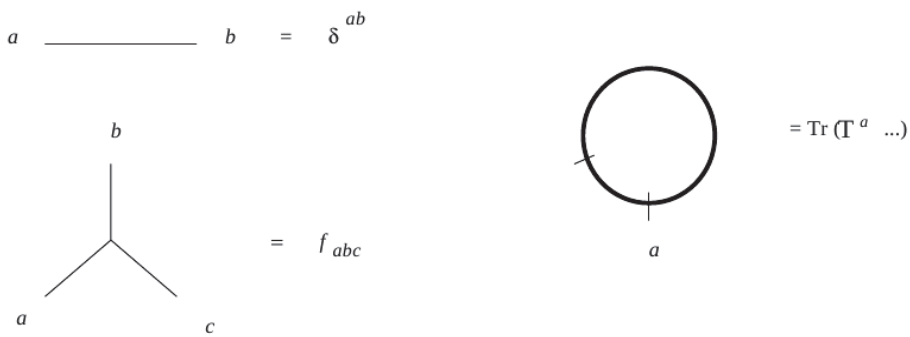
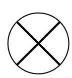
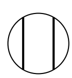



Fig. 2. Group-theoretical rules.

Let us explain the definition of trivalent diagrams on the first relation from Fig. 1:

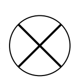
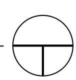
$$T^aT^bT^cT^d\delta^{ac}\delta^{bd} = T^aT^bT^aT^b =$$  ,

$$T^aT^bT^cT^d\delta^{ad}\delta^{bc} = T^aT^bT^bT^a =$$ 

$$= T^aT^bT^aT^b - T^aT^bT^aT^b + T^aT^bT^bT^a$$

$$= T^aT^bT^aT^b - T^aT^b(T^aT^b - T^bT^a)$$

$$= T^aT^bT^aT^b - T^aT^b[T^aT^b]$$

$$= T^aT^bT^aT^b - f^{abc}T^aT^bT^c =$$  $-$  .

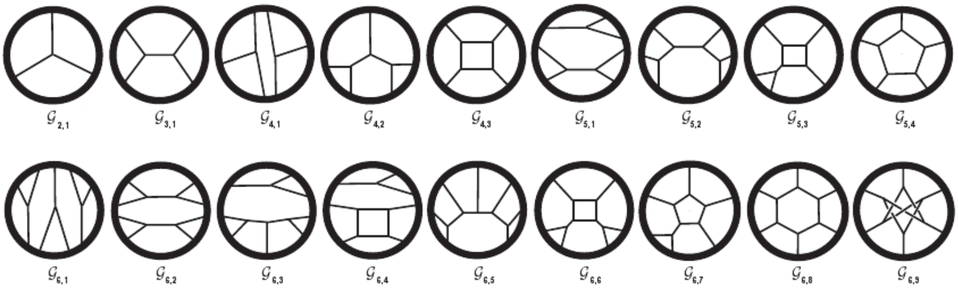



Fig. 3. Trivalent diagrams.

In Fig. 3, one can find a collection of trivalent diagrams that form the so-called *canonical basis* $\{\mathcal{G}_{ij}\}$ of $H_n$ up to order six. In the fundamental representation $R[1]$, their explicit expressions are given in the following table:

$$
\begin{aligned}
&\mathcal{G}_{2,1} = -\frac{1}{4}N^2 + \frac{1}{4}, && \mathcal{G}_{6,1} = -\frac{1}{64}N^6 + \frac{3}{64}N^4 - \frac{3}{64}N^2 + \frac{1}{64},\\
&\mathcal{G}_{3,1} = -\frac{1}{8}N^3 + \frac{1}{8}N, && \mathcal{G}_{6,2} = \frac{1}{64}N^6 - \frac{1}{32}N^4 + \frac{1}{64}N^2,\\
&\mathcal{G}_{4,1} = \frac{1}{16}N^4 - \frac{1}{8}N^2 + \frac{1}{16}, && \mathcal{G}_{6,3} = \frac{1}{64}N^6 - \frac{1}{32}N^4 + \frac{1}{64}N^2,\\
&\mathcal{G}_{4,2} = -\frac{1}{16}N^4 + \frac{1}{16}N^2, && \mathcal{G}_{6,4} = -\frac{1}{64}N^6 + \frac{3}{64}N^2 - \frac{1}{32},\\
&\mathcal{G}_{4,3} = \frac{1}{16}N^4 + \frac{1}{16}N^2 - \frac{1}{8}, && \mathcal{G}_{6,5} = -\frac{1}{64}N^6 + \frac{1}{64}N^4,\\
&\mathcal{G}_{5,1} = \frac{1}{32}N^5 - \frac{1}{16}N^3 + \frac{1}{32}N, && \mathcal{G}_{6,6} = \frac{1}{64}N^6 + \frac{1}{64}N^4 - \frac{1}{32}N^2,\\
&\mathcal{G}_{5,2} = -\frac{1}{32}N^5 + \frac{1}{32}N^3, && \mathcal{G}_{6,7} = \frac{1}{64}N^6 - \frac{1}{64}N^2,\\
&\mathcal{G}_{5,3} = \frac{1}{32}N^5 + \frac{1}{32}N^3 - \frac{1}{16}N, && \mathcal{G}_{6,8} = \frac{1}{64}N^6 + \frac{1}{64}N^2 - \frac{1}{32},\\
&\mathcal{G}_{5,4} = \frac{1}{32}N^5 - \frac{1}{32}N, && \mathcal{G}_{6,9} = \frac{3}{64}N^4 - \frac{5}{64}N^2 + \frac{1}{32}.
\end{aligned}
\tag{5}
$$

In the first symmetric representation $R = [2]$ they equal to

$$
\begin{aligned}
&\mathcal{G}_{2,1} = -\frac{1}{2}N^2 - \frac{1}{2}N + 1, && \mathcal{G}_{6,1} = \left(\mathcal{G}_{2,1}\right)^3,\\
&\mathcal{G}_{3,1} = -\frac{1}{4}N^3 - \frac{1}{4}N^2 + \frac{1}{2}N, && \mathcal{G}_{6,2} = \left(\mathcal{G}_{3,1}\right)^2,\\
&\mathcal{G}_{4,1} = \left(\mathcal{G}_{2,1}\right)^2, && \mathcal{G}_{6,3} = \mathcal{G}_{2,1} \cdot \mathcal{G}_{4,2},\\
&\mathcal{G}_{4,2} = -\frac{1}{8}N^4 - \frac{1}{8}N^3 + \frac{1}{4}N^2, && \mathcal{G}_{6,4} = \mathcal{G}_{2,1} \cdot \mathcal{G}_{4,3},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{G}_{4,3} &= \frac{1}{8}N^4 + \frac{3}{8}N^3 + N^2 - \frac{1}{8}N - 2\,,\\
\mathcal{G}_{6,5} &= -\frac{1}{32}N^6 - \frac{1}{32}N^5 + \frac{1}{16}\,,\\
\mathcal{G}_{5,1} &= \mathcal{G}_{2,1}\cdot\mathcal{G}_{3,1}\,,\\
\mathcal{G}_{6,6} &= \frac{1}{32}N^6 + \frac{3}{32}N^5 + \frac{1}{4}N^4 + \frac{1}{8}N^3 - \frac{1}{2}N^2\,,\\
\mathcal{G}_{5,2} &= -\frac{1}{16}N^5 - \frac{1}{16}N^4 + \frac{1}{8}N^3\,,\\
\mathcal{G}_{6,7} &= \frac{1}{32}N^6 + \frac{1}{8}N^5 + \frac{11}{32}N^4 + \frac{1}{8}N^3 - \frac{5}{8}N^2\,,\\
\mathcal{G}_{5,3} &= \frac{1}{16}N^5 + \frac{3}{16}N^4 + \frac{1}{2}N^3 + \frac{1}{4}N^2 + N\,,\\
\mathcal{G}_{6,8} &= \frac{1}{32}N^6 + \frac{5}{32}N^5 + \frac{19}{32}N^4 + \frac{31}{32}N^3 + \frac{5}{8}N^2 - \frac{3}{8}N - 2\,,\\
\mathcal{G}_{5,4} &= -\frac{1}{16}N^5 - \frac{1}{4}N^4 - \frac{11}{16}N^3 - \frac{1}{4}N^2 + \frac{5}{4}N\,,\\
\mathcal{G}_{6,9} &= \frac{7}{32}N^4 + \frac{9}{32}N^3 - \frac{11}{8}N^2 - \frac{9}{8}N + 2\,.
\end{aligned}
\tag{6}
$$

Using this basis, we rewrite (3) through invariants:

$$
\langle W_R(K)\rangle = \sum_{n=0}^{\infty}\hbar^n \sum_{m=1}^{\dim(H_n)} \mathcal{V}_{n,m}\mathcal{G}_{n,m}\,. \tag{7}
$$

Here, $\mathcal{V}_{ij}$ are the so-called finite-type or Vassiliev invariants of knots. They depend only on the knot under consideration but not on the group and its representation.

Now, let us introduce primitive Vassiliev invariants. It is a well-known fact that the expansion of logarithm of any correlator in any QFT contains only connected Feynman diagrams (for more details about this situation in the Chern–Simons perturbation theory see Ref. 27). This fact immediately leads to the following summation of

$$
\langle W_R(K)\rangle = \prod_{n=0}^{\infty}\prod_{m=1}^{\mathcal{N}_n} \exp\bigl(\hbar^n \mathcal{V}^c_{n,m}\mathcal{G}^c_{n,m}\bigr)\,, \tag{8}
$$

where $\mathcal{G}^c$ are connected diagrams, $\mathcal{V}^c$ are primitive Vassiliev invariants. The Vassiliev invariants form a graded ring freely generated by primitive invariants. Here, $\mathcal{N}_n$ is dimension of the space of connected chord diagrams (or equivalently the space of primitive Vassiliev invariants). The dimensions of these spaces up to order 6 are given in the following table:

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $\mathcal{N}_n$ | 1 | 1 | 1 | 2 | 3 | 5 |

(9)

The meaning of the expression (8) is that $\mathcal{V}_{i,j}$ in (7) are not independent. In fact only those coefficients $\mathcal{V}_{ij}$ are independent, for which the corresponding diagram $\mathcal{G}_{ij}$ is connected. Comparing $\hbar$ expansion of (8) with (7) we, for example, find:

$$\begin{aligned} \mathcal{V}_{4,1} &= \frac{1}{2}\,\mathcal{V}_{2,1}^2\,, & \mathcal{V}_{5,1} &= \mathcal{V}_{2,1}\mathcal{V}_{3,1}\,, \\ \mathcal{V}_{6,1} &= \frac{1}{6}\mathcal{V}_{2,1}^3\,, & \mathcal{V}_{6,2} &= \frac{1}{2}\,\mathcal{V}_{3,1}^2\,, \\ \mathcal{V}_{6,3} &= \mathcal{V}_{2,1}\mathcal{V}_{4,2}\,, & \mathcal{V}_{6,4} &= \mathcal{V}_{2,1}\mathcal{V}_{4,3}\,. \end{aligned} \tag{10}$$

Last but not least, from formulas (6) we can see that to compute Vassiliev invariants up to order 6 it is enough to have HOMFLY polynomials in the first symmetric representation $R = [2]$. In the next section, we provide corresponding formulas of HOMFLY polynomials for torus and pretzel knots.

## 3. HOMFLY Polynomials

### 3.1. *Torus knots*

In the case of torus knots, HOMFLY polynomials in all representations were calculated by Rosso and Jones in Refs. 19 and 20. So, let us consider torus knot $T[m,n]$ with mutually prime $m$ and $n$ and let $\chi_R$ are the Schur polynomials. We define the coefficients $c_R^Q$ from the relation

$$\chi_R\{p^{(m)}\} = \sum_{Q\vdash m|R|} c_R^Q \chi_Q\{p\}\,, \tag{11}$$

where

$$p_k^{(m)} = p_{mk}\,. \tag{12}$$

Thus, for the torus knot $T[m,n]$ one has

$$H_R^{T[m,n]}\{p\} = \sum_{Q\vdash m|R|} q^{-2\frac{n}{m}\varkappa_Q} c_R^Q \chi_Q^*\,, \tag{13}$$

where

$$\chi_Q^* = \chi_Q\left\{p_k = \frac{a^k - a^{-k}}{q^k - q^{-k}}\right\}. \tag{14}$$

This nice formula allows to compute HOMFLY polynomial in any representation.

### 3.2. *Pretzel knots*

These are knots and links formed by wrapping around a surface of genus $g$ without self-intersections, which can be different from $g = 1$. The simplest set of this type has a knot diagram (see Fig. 2), consisting of $g+1$ two-strand braids, and thus has $g+1$ different parameters $n_1, \ldots, n_{g+1}$ (for $g = 1$ everything depends on the sum

$n = n_1 + n_2$). In the literature (see Ref. 28) this family is known as the pretzel knots and links. The family is actually split into subfamilies, differing by mutual orientation of strands in the braids. Since we are interested in pretzel knots only, let us consider all possible configurations of parameters $n_1, \dots, n_{g+1}$ and orientations, which provide only knots. There are three possible orientations: *antiparallel*, *parallel* and *mixed*.

**Antiparallel.** In the first case, we put genus $g$ to be odd, all orientations of constituent braids must be antiparallel like on the picture  all parameters $\overline{n_1}, \dots, \overline{n_g}$ must be odd.[a] Then, we obtain a class of knots which we refer to *antiparallel* pretzel knots. Their HOMFLY polynomials in any symmetric representation are given as follows:

$$H_R^{\overline{n_1},\dots,\overline{n_{g+1}}} = \sum_{k=0}^{r} \Delta_k \prod_{i=1}^{g+1} \sum_{m=0}^{r} \bar{\bar{a}}_{km} \bar{\lambda}_m^{\bar{n}_i}\,, \tag{15}$$

where $\Delta_k$ is *a quantum dimension* of the corresponding representation, $\bar{\lambda}_m$ is *an eigenvalue* of the corresponding R-matrix and $\bar{\bar{a}}_{km}$ is the corresponding *Racah coefficient.* Their values were computed in Refs. 22 and 23 and can be listed as follows:

$$\begin{aligned}
\bar{\lambda}_m &= \left(-q^{m-1}A\right)^m\,,\\
\Delta_m &= \chi^*_{[r+m,r-m]} = \frac{[2m+1]}{[r+m+1]![r-m]!} \prod_{i=0}^{2r-1} D_j \prod_{j=0}^{r-m-1} \frac{D_{j-1}}{D_{r+m+j}}\,,\\
\bar{\Delta}_m &= D_{2m-1} \cdot \left( \prod_{j=0}^{m-2} \frac{D_j}{[j+2]} \right)^2 \cdot D_{-1}\,,\\
\bar{\bar{a}}_{km} &= \frac{\bar{\Delta}_m}{\Delta_k} a_{mk}\,, \quad a_{km} = \alpha_{km} \cdot \mathcal{G}\,,\\
\alpha_{km} &= (-1)^{r+k+m}[2m+1] \cdot \frac{([k]![m]!)^2[r-k]!\,[r-m]!}{[r+k+1]!\,[r+m+1]!} \sum_{j=\max(r+m,r+k)}^{\min(r+k+m,2r)}\\
&\quad \times \frac{(-1)^j\,[j+1]!}{[2r-j]!\big([j-r-k]!\,[j-r-m]!\,[r+k+m-j]!\big)^2}\,,\\
\mathcal{G} &= \frac{G(r-m)G(j+1)}{G(r+k+1)G(j-r-m)}\,,\\
G(n) &= \frac{1}{[n]!} \prod_{i=-1}^{n-2} D_i = \frac{(A/q;q)_n}{(q;q)_n}
\end{aligned} \tag{16}$$

[a]Since for all qualities standing for the antiparallel case we use "bar," we denote parameters in this case as $\overline{n_1}, \dots, \overline{n_{g+1}}$.

and also we used here standard notations for quantum numbers $\{x\} = x - x^{-1}$, $[x] = \frac{\{q^x\}}{\{q\}}$ and quantum factorials, for differentials $D_i = \frac{\{Aq^i\}}{\{q\}}$ and for symmetric $q$-Pochhammer symbols $(A;q)_n = \prod_{j=0}^{n-1}\{Aq^j\}$. Also note that at $A = q^N$, $G(n)$ becomes the $q$-binomial $\binom{N+n-2}{n}_q$.

**Parallel.** In the second case, we put genus $g$ to be odd, all orientations of constituent braids must be parallel like on the picture  all parameters $n_2, \ldots, n_{g+1}$ must be odd and $n_1$ must be even.[b] Then, we obtain a class of knots which we refer to *parallel* pretzel knots. Their HOMFLY polynomials in any symmetric representation are given as follows:

$$H_{[r]}^{n_1,\ldots,n_{g+1}} = \sum_{k=0}^{r} \bar{\Delta}_k \cdot \left\{ \prod_{i=1}^{g+1} \left( \sum_{m=0}^{r} a_{km} \lambda_m^{n_i} \right) \right\}, \tag{17}$$

where all constituents have similar meaning as in the previous case, explicit expression for $\lambda_m$ is the following

$$\lambda_m = (-)^{m+1} \frac{q^{-r^2+m^2+m}}{A^r}. \tag{18}$$

**Mixed.** In the third case, we put genus $g$ to be even, all orientations of constituent braids, except one, must be parallel, their corresponding parameters $n_2, \ldots, n_{g+1}$ must be odd, $\bar{n}_1$ must be even (again for simplicity distinguish $n_1$). Then, we obtain a class of knots which we refer to *mixed* pretzel knots. Their HOMFLY polynomials in any symmetric representation are given as follows:

$$H_{[r]}^{\bar{n}_1,n_2,\ldots,n_{g+1}} = \sum_{k=0}^{r} \bar{\Delta}_k \cdot \left\{ \left( \sum_{m=0}^{r} \bar{a}_{km} \bar{\lambda}_m^{\bar{n}_1} \right) \prod_{i=2}^{g+1} \left( \sum_{m=0}^{r} a_{km} \lambda_m^{n_i} \right) \right\}. \tag{19}$$

Thus, our formulas (15), (17) and (19) provide the explicit answer for arbitrary pretzel knots in arbitrary symmetric representation. These formulas we can use to evaluate Vassiliev invariants in the next section.

## 4. Vassiliev Invariants

In this section, we first present the Vassiliev invariants up to order 6 for torus knots evaluated by Alvarez and Labastida in Ref. 21, then we present our results for pretzel knots.

[b] We choose $n_1$ to be even for simplicity. In general, it is possible to choose any.

### 4.1. *Torus knots*

$$\begin{aligned} v_{2,1} &= \frac{1}{24}(n^2-1)(m^2-1)\,, \\ v_{3,1} &= \frac{1}{144}nm(n^2-1)(m^2-1)\,, \\ v_{4,2} &= \frac{1}{240}(n^2-1)(m^2-1)(9n^2m^2-n^2-m^2-1)\,, \\ v_{4,3} &= \frac{1}{240}(n^4-1)(m^4-1)\,, \end{aligned} \tag{20}$$

$$\begin{aligned} v_{5,2} &= \frac{1}{28800}nm(n^2-1)(m^2-1)(69n^2m^2-21\big(n^2+m^2\big)-11\big)\,, \\ v_{5,3} &= \frac{1}{57600}nm(n^2-1)(m^2-1)(11n^2m^2+n^2+m^2-9)\,, \\ v_{5,4} &= \frac{1}{7200}nm(n^4-1)(m^4-1)\,, \end{aligned} \tag{21}$$

$$\begin{aligned} v_{6,5} &= \frac{1}{2520}(n^2-1)(m^2-1)\big(516n^4m^4-289(n^2m^4+n^4m^2) \\ &\quad -44n^2m^2+5\big(n^4+m^4\big)+5(n^2+m^2)+5\big)\,, \\ v_{6,6} &= \frac{1}{12096}(n^2-1)(m^2-1)\big(53n^4m^4-101(n^2m^4+n^4m^2) \\ &\quad -115n^2m^2-24\big(n^4+m^4\big)-24(n^2+m^2)-24\big)\,, \\ v_{6,7} &= \frac{1}{10080}(n^2-1)(m^2-1)\big(419n^4m^4+209\big(n^2m^4+n^4m^2\big) \\ &\quad -n^2m^2+20\big(n^4+m^4\big)+20\big(n^2+m^2\big)+20\big)\,, \\ v_{6,8} &= \frac{1}{25200}(n^2-1)(m^2-1)\big(13n^4m^4+13\big(n^2m^4+n^4m^2\big) \\ &\quad +13n^2m^2-50\big(n^4+m^4\big)-50\big(n^2+m^2\big)-50\big)\,, \\ v_{6,9} &= \frac{1}{5040}(n^2-1)(m^2-1)\big(31n^4m^4+31\big(n^2m^4+n^4m^2\big) \\ &\quad +31n^2m^2+10\big(n^4+m^4\big)+10\big(n^2+m^2\big)+10\big)\,. \end{aligned} \tag{22}$$

### 4.2. *Pretzel knots*

HOMFLY polynomials, obtained in the previous section, are symmetric under permutations of $\{n_i\}$. The reason is the following. Permutation of the two adjacent $n_i$'s is just a knot mutation. Since the HOMFLY polynomials in symmetric representations do not distinguish mutant knots,[29] then with the help of mutation, one can permute $n_i \leftrightarrow n_{i+1}$. Vassiliev invariants up to order 6 also do not distinguish the mutant knots, hence their formulas have to include this symmetry. Taking this into account together with the polynomiality in $\{n_i\}$, we conclude that Vassiliev invariants can be expressed in terms of symmetric polynomials. In other words, we can choose some basis in the space of symmetric polynomials and express Vassiliev invariants in terms of basis elements with some coefficients depending on genus $g$. Schur polynomials provide a distinguished basis in the space of symmetric polynomials, so we use it for our computations.

Below we present our results for three subfamilies of pretzel knots. To avoid notation ambiguities, we use different letters standing for Vassiliev invariants for different subfamilies.

4.2.1. *Antiparallel*

$$v_{2,1} = \chi_{[1,1]} + \frac{\mathrm{g}}{2} = \frac{1}{2}\left(\sum_{i=1}^{g+1} n_i\right)^2 - \frac{1}{2}\sum_{i=1}^{g+1} n_i^2 + \frac{\mathrm{g}}{2}\,, \tag{23}$$

$$v_{3,1} = -\frac{1}{2}\left(\chi_{[2,1]} + 2\chi_{[1,1,1]} + \mathrm{g}\chi_{[1]}\right), \tag{24}$$

$$\begin{aligned} v_{4,2} = \frac{1}{6}\bigg(&\chi_{[3,1]} + 5\chi_{[2,2]} + 8\chi_{[2,1,1]} + 5\chi_{[1,1,1,1]} \\ &+ \frac{3}{2}\mathrm{g}\chi_{[2]} + (9\mathrm{g}+4)\frac{1}{2}\chi_{[1,1]} + \frac{\mathrm{g}}{4}(3\mathrm{g}+2)\bigg), \end{aligned} \tag{25}$$

$$v_{4,3} = \frac{1}{12}\left(3\chi_{[2,2]} + 3\chi_{[2,1,1]} - 3\chi_{[1,1,1,1]} + 4\chi_{[1,1]} + \frac{\mathrm{g}}{2}\right), \tag{26}$$

$$\begin{aligned} v_{5,2} = -\frac{1}{24}\big(&\chi_{[4,1]} + 21\chi_{[3,2]} + 25\chi_{[3,1,1]} + 63\chi_{[2,2,1]} \\ &+ 57\chi_{[2,1,1,1]} + 14\chi_{[1,1,1,1,1]} + 2\mathrm{g}\chi_{[3]} \\ &+ 4(2+7\mathrm{g})\chi_{[2,1]} + 2(12+13\mathrm{g})\chi_{[1,1,1]} + \mathrm{g}(6\mathrm{g}+5)\chi_{[1]}\big), \end{aligned} \tag{27}$$

$$\begin{aligned} v_{5,3} = -\frac{1}{6}\bigg(&\chi_{[3,2]} + \chi_{[3,1,1]} + 3\chi_{[2,2,1]} + 3\chi_{[2,1,1,1]} \\ &- 2\chi_{[1,1,1,1,1]} + \frac{3}{2}\mathrm{g}\chi_{[2,1]} + 4\chi_{[1,1,1]} + \mathrm{g}\chi_{[1]}\bigg), \end{aligned} \tag{28}$$

$$v_{5,4} = -\frac{1}{6}\left(\chi_{[3,2]} + \chi_{[3,1,1]} + 3\chi_{[2,2,1]} - 2\chi_{[1,1,1,1,1]} + 2\chi_{[2,1]} + 2\chi_{[1,1,1]}\right), \tag{29}$$

$$\begin{aligned} v_{6,5} = \frac{1}{360}\bigg(&3\chi_{[5,1]} + 192\chi_{[4,2]} + 405\chi_{[3,3]} + 207\chi_{[4,1,1]} \\ &+ 1578\chi_{[3,2,1]} + 1113\chi_{[3,1,1,1]} + 1050\chi_{[2,2,2]} + 2097\chi_{[2,2,1,1]} \\ &+ 1332\chi_{[2,1,1,1,1]} + 81\chi_{[1,1,1,1,1,1]} + \frac{15}{2}\mathrm{g}\chi_{[4]} \end{aligned}$$

$$+\frac{15}{2}(47\mathrm{g}+8)\chi_{[3,1]}+90(7\mathrm{g}+2)\chi_{[2,2]}+\frac{15}{2}(145\mathrm{g}+64)\chi_{[2,1,1]}$$
$$+\frac{15}{2}(71\mathrm{g}+104)\chi_{[1,1,1,1]}+\frac{15}{2}g(15\mathrm{g}+8)\chi_{[2]}$$
$$+\frac{3}{2}(135\mathrm{g}^2+170\mathrm{g}-32)\chi_{[1,1]}+\frac{3}{2}\mathrm{g}(10\mathrm{g}^2+15\mathrm{g}+6)\Big)\,, \tag{30}$$

$$v_{6,6}=\frac{1}{360}\Big(-35\chi_{[3,3]}-70\chi_{[3,2,1]}+85\chi_{[3,1,1,1]}-50\chi_{[2,2,2]}+225\chi_{[2,2,1,1]}$$
$$+470\chi_{[2,1,1,1,1]}-80\chi_{[1,1,1,1,1,1]}+20(3\mathrm{g}-4)\chi_{[3,1]}$$
$$+20(6\mathrm{g}-11)\chi_{[2,2]}+10(21\mathrm{g}-31)\chi_{[2,1,1]}+10(3\mathrm{g}+41)\chi_{[1,1,1,1]}$$
$$+\frac{15}{2}\mathrm{g}(3\mathrm{g}+1)\chi_{[2]}+\frac{3}{2}(15\mathrm{g}^2+85\mathrm{g}-98)\chi_{[1,1]}+3\mathrm{g}(5\mathrm{g}+3)\Big)\,, \tag{31}$$

$$v_{6,7}=\frac{1}{360}\big(75\chi_{[4,2]}+255\chi_{[3,3]}+75\chi_{[4,1,1]}+810\chi_{[3,2,1]}+375\chi_{[3,1,1,1]}$$
$$+525\chi_{[2,2,2]}+585\chi_{[2,2,1,1]}-210\chi_{[2,1,1,1,1]}-330\chi_{[1,1,1,1,1,1]}$$
$$+60(\mathrm{g}+2)\chi_{[3,1]}+60(2\mathrm{g}+7)\chi_{[2,2]}+60(2\mathrm{g}+13)\chi_{[2,1,1]}$$
$$-60(\mathrm{g}-5)\chi_{[1,1,1,1]}+45\mathrm{g}\chi_{[2]}+(105\mathrm{g}+124)\chi_{[1,1]}+2\mathrm{g}\big)\,, \tag{32}$$

$$v_{6,8}=\frac{1}{144}\big(3\chi_{[4,2]}-5\chi_{[3,3]}+3\chi_{[4,1,1]}+2\chi_{[3,2,1]}-5\chi_{[3,1,1,1]}$$
$$+7\chi_{[2,2,2]}+15\chi_{[2,2,1,1]}-4\chi_{[2,1,1,1,1]}-20\chi_{[1,1,1,1,1,1]}$$
$$-8\chi_{[2,2]}+8\chi_{[3,1]}+4\chi_{[2,1,1]}+28\chi_{[1,1,1,1]}-10\chi_{[1,1]}-\mathrm{g}\big)\,, \tag{33}$$

$$v_{6,9}=\frac{1}{720}\big(15\chi_{[4,2]}+95\chi_{[3,3]}+15\chi_{[4,1,1]}+250\chi_{[3,2,1]}$$
$$+95\chi_{[3,1,1,1]}+155\chi_{[2,2,2]}+75\chi_{[2,2,1,1]}$$
$$-140\chi_{[2,1,1,1,1]}+20\chi_{[1,1,1,1,1,1]}+40\chi_{[3,1]}$$
$$+200\chi_{[2,2]}+260\chi_{[2,1,1]}-100\chi_{[1,1,1,1]}+86\chi_{[1,1]}+3\mathrm{g}\big)\,. \tag{34}$$

Let us note that any $\chi_\Delta$ depends on genus $g$, because it depends on $g+1$ variables $\{n_1,\dots,n_{g+1}\}$. However, some coefficients of $\chi_\Delta$ in the formulas above do not depend on $g$. Actually, we can make the following three observations:

(1) Coefficients of leading terms do not depend on $g$, i.e. they are constants;
(2) Coefficients in $v_{5,4}$ are constants;
(3) Three following combinations have constant coefficients:

$$120v_{6,9}-v_{2,1}\,,\quad 12v_{4,3}-v_{2,1}\,,\quad 72v_{6,8}+v_{2,1}\,. \tag{35}$$

4.2.2. *Parallel*

$$u_{2,1} = \big(n_1^2 + 2\chi_{[1]}n_1 + \chi_{[2]} - \chi_{[1,1]} - \mathrm{g}\big)\frac{1}{2}\,, \tag{36}$$

$$\begin{aligned} u_{3,1} = -\Bigg(&n_1^3 + 3\chi_{[1]}n_1^2 + n_1\left(3\chi_{[2]} - (3\mathrm{g}-1)\frac{1}{2}\right) \\ &+ \big(\chi_{[3]} - \chi_{[2,1]} + \chi_{[1,1,1]} - \chi_{[1]}\big)\Bigg)\frac{1}{3}\,, \end{aligned} \tag{37}$$

$$\begin{aligned} u_{4,2} = \Bigg(&\frac{14}{3}\big(\chi_{[4]} - \chi_{[3,1]} + \chi_{[2,1,1]} - \chi_{[1,1,1,1]}\big) - \frac{16}{3}\big(\chi_{[2]} - \chi_{[1,1]}\big) \\ &+ \frac{2\mathrm{g}}{3} + n_1 \cdot \frac{8}{3}\big(7\chi_{[3]} - \chi_{[2,1]} + \chi_{[1,1,1]} - (3\mathrm{g}+1)\chi_{[1]}\big) \\ &+ n_1^2 \cdot \frac{4}{3}\big(21\chi_{[2]} + 9\chi_{[1,1]} - 2(3\mathrm{g}-1)\big) \\ &+ n_1^3 \cdot \frac{56}{3}\chi_{[1]}n_1^3 + \frac{14}{3}n_1^4\Bigg)\frac{1}{16}\,, \end{aligned} \tag{38}$$

$$\begin{aligned} u_{4,3} = \Bigg(&\frac{2}{3}\big(\chi_{[4]} - \chi_{[3,1]} + \chi_{[2,1,1]} - \chi_{[1,1,1,1]}\big) \\ &- \frac{2\mathrm{g}}{3} + \frac{8}{3}\big(\chi_{[3]} - \chi_{[2,1]} + \chi_{[1,1,1]}\big)n_1 \\ &+ n_1^2 \cdot 4\big(\chi_{[2]} + \chi_{[1,1]}\big) + n_1^3 \cdot \frac{8}{3}\chi_{[1]} + n_1^4 \cdot \frac{2}{3}\Bigg)\frac{1}{16}\,, \end{aligned} \tag{39}$$

$$\begin{aligned} u_{5,2} = \Bigg(&-51\big(\chi_{[5]} - \chi_{[4,1]} + \chi_{[3,1,1]} - \chi_{[2,1,1,1]} + \chi_{[1,1,1,1,1]}\big) \\ &+ 70\big(\chi_{[3]} - \chi_{[2,1]} + \chi_{[1,1,1]}\big) - 19\chi_{[1]} \\ &+ n_1 \cdot \Bigg(-255\chi_{[4]} + 45\chi_{[3,1]} + 30\chi_{[2,2]} - 45\chi_{[2,1,1]} \\ &+ 15\chi_{[1^4]} + 30(4+3\mathrm{g})\chi_{[2]} - 30\chi_{[1,1]} - \left(\frac{45}{2}\mathrm{g}^2 - 15\mathrm{g} + \frac{23}{2}\right)\Bigg) \\ &+ n_1^2\big(-510\chi_{[3]} - 255\chi_{[2,1]} - 60\chi_{[1,1,1]} + 15(-1+15\mathrm{g})\chi_{[1]}\big) \\ &+ n_1^3\big(-510\chi_{[2]} - 300\chi_{[1,1]} + 35(-1+3\mathrm{g})\big) \\ &+ n_1^4\big(-255\chi_{[1]}\big) - 51n_1^5\Bigg)\frac{1}{180}\,, \end{aligned} \tag{40}$$

$$\begin{aligned} u_{5,3} = \big(&-9\big(\chi_{[5]} - \chi_{[4,1]} + \chi_{[3,1,1]} - \chi_{[2,1,1,1]} + \chi_{[1,1,1,1,1]}\big) \\ &+ 10\big(\chi_{[3]} - \chi_{[2,1]} + \chi_{[1,1,1]}\big) - \chi_{[1]} \\ &+ n_1 \cdot \big(-45\chi_{[4]} + 15\chi_{[3,1]} + 30\chi_{[2,2]} - 15\chi_{[2,1,1]} \\ &- 15\chi_{[1,1,1,1]} + 30\chi_{[2]} + 30\chi_{[1,1]} + (-16+15\mathrm{g})\big) \end{aligned}$$

$$+ n_1^2 \cdot \big(-90\chi_{[3]} - 45\chi_{[2,1]} + 15(-1+3\mathrm{g})\chi_{[1]}\big)$$
$$+ n_1^3 \cdot \big(-90\chi_{[2]} - 60\chi_{[1,1]} + 5(-1+3\mathrm{g})\big)$$
$$+ n_1^4 \cdot (-45\chi_{[1]}) - 9 \cdot n_1^5\big)\frac{1}{180}\,, \tag{41}$$

$$u_{5,4} = -\big(\chi_{[5]} - \chi_{[4,1]} + \chi_{[3,1,1]} - \chi_{[2,1^3]} + \chi_{[1^5]} - \chi_{[1]}$$
$$+ n_1 \cdot \big(5\big(\chi_{[4]} - \chi_{[3,1]} + \chi_{[2,1,1]} - \chi_{[1^4]}\big) + 10\chi_{[1,1]} - 1\big)$$
$$+ n_1^2 \cdot \big(10\chi_{[3]} + 5\chi_{[2,1]} + 10\chi_{[1,1,1]}\big)$$
$$+ n_1^3 \cdot \big(10\chi_{[2]} + 10\chi_{[1,1]}\big) + n_1^4 \cdot \big(5\chi_{[1]}\big) + n_1^5\big)\frac{1}{30}\,, \tag{42}$$

$$u_{6,5} = \big(203\big(\chi_{[6]} - \chi_{[5,1]} + \chi_{[4,1,1]} - \chi_{[3,1^3]} + \chi_{[2,1^4]} - \chi_{[1^6]}\big)$$
$$- 340\big(\chi_{[4]} - \chi_{[3,1]} + \chi_{[2,1,1]} - \chi_{[1^4]}\big) + 140\big(\chi_{[2]} - \chi_{[1,1]}\big)$$
$$- 3\mathrm{g} + n_1 \cdot \big(1218\chi_{[5]} - 198\chi_{[4,1]} - 180\chi_{[3,2]} + 198\chi_{[3,1,1]}$$
$$+ 60\chi_{[2,2,1]} - 78\chi_{[2,1^3]} + 18\chi_{[1^3]} - 20(47+21\mathrm{g})\chi_{[3]}$$
$$+ 20(11+3\mathrm{g})\chi_{[2,1]} - 20(5+3\mathrm{g})\chi_{[1,1,1]} + 10(9\mathrm{g}^2+6\mathrm{g}+13)\chi_{[1]}\big)$$
$$+ n_1^2 \cdot \big(3045\chi_{[4]} + 1635\chi_{[3,1]} + 360\chi_{[2,2]} + 345\chi_{[2,1,1]} + 15\chi_{[1,1,1,1]}$$
$$- 510(1+3\mathrm{g})\chi_{[2]} - 30(-1+21\mathrm{g})\chi_{[1,1]} + 5(45\mathrm{g}^2 - 33\mathrm{g} + 16)\big)$$
$$+ n_1^3 \cdot \big(4060\chi_{[3]} + 3440\chi_{[2,1]} + 820\chi_{[1,1,1]} - 20\big(-13+81\mathrm{g}\big)\chi_{[1]}\big)$$
$$+ n_1^4 \cdot \big(3045\chi_{[2]} + 2025\chi_{[1,1]} - 170(-1+3\mathrm{g})\big)$$
$$+ 1218\chi_{[1]} \cdot n_1^5 + 203 \cdot n_1^6\big)\frac{1}{720}\,, \tag{43}$$

$$u_{6,6} = \Bigg(5\big(\chi_{[6]} - \chi_{[5,1]} + \chi_{[4,1,1]} - \chi_{[3,1^3]} + \chi_{[2,1^4]} - \chi_{[1^6]}\big)$$
$$- 30\big(\chi_{[4]} - \chi_{[3,1]} + \chi_{[2,1,1]} - \chi_{[1^4]}\big) + 19\big(\chi_{[2]} - \chi_{[1,1]}\big) + 6\mathrm{g}$$
$$+ n_1 \cdot \big(30\chi_{[5]} + 60\chi_{[4,1]} - 60\chi_{[3,1,1]} - 30\chi_{[3,2]}$$
$$- 30\chi_{[2,2,1]} + 120\chi_{[2,1,1,1]} - 90\chi_{[1,1,1,1,1]} - 30(3+\mathrm{g})\chi_{[3]}$$
$$- -30(6-\mathrm{g})\chi_{[2,1]} - 30(-1+\mathrm{g})\chi_{[1,1,1]} + 2(4+15\mathrm{g})\chi_{[1]}\big)$$
$$+ n_1^2 \cdot \Bigg(75\chi_{[4]} + 120\chi_{[3,1]} + 30\chi_{[2,2]} - 165\chi_{[2,1,1]}$$
$$+ (45\mathrm{g}^2 - 75\mathrm{g} + 68)\frac{1}{2} - 45(1+3\mathrm{g})\chi_{[2]} - 45(5+\mathrm{g})\chi_{[1,1]}\Bigg)$$
$$+ n_1^3 \cdot \big(100\chi_{[3]} + 5\chi_{[2,1]} - 200\chi_{[1,1,1]} - 30(-1+5\mathrm{g})\chi_{[1]}\big)$$
$$+ n_1^4 \cdot \big(75\chi_{[2]} - 15\chi_{[1,1]} - 15(-1+3\mathrm{g})\big) + 30\chi_{[1]} \cdot n_1^5 + 5 \cdot n_1^6\Bigg)\frac{1}{360}\,, \tag{44}$$

$$u_{6,7} = \Big(18\big(\chi_{[6]} - \chi_{[5,1]} + \chi_{[4,1,1]} - \chi_{[3,1^3]} + \chi_{[2,1^4]} - \chi_{[1^6]}\big)$$
$$- 10\big(\chi_{[4]} - \chi_{[3,1]} + \chi_{[2,1,1]} - \chi_{[1^4]}\big) - 7\chi_{[2]} + 7\chi_{[1,1]} - \mathrm{g}$$
$$+ n_1 \cdot \big(108\chi_{[5]} - 78\chi_{[4,1]} + 78\chi_{[3,1,1]} - 30\chi_{[3,2]}$$
$$+ 30\chi_{[2,2,1]} - 78\chi_{[2,1,1,1]} + 48\chi_{[1,1,1,1,1]} - 40\chi_{[3]}$$
$$+ 130\chi_{[2,1]} - 40\chi_{[1,1,1]} - 2(-8 + 15\mathrm{g})\chi_{[1]}\big)$$
$$+ n_1^2 \cdot \Big(270\chi_{[4]} + 75\chi_{[3,1]} - 30\chi_{[2,2]} + 150\chi_{[2,1,1]}$$
$$+ 15\chi_{[1^4]} - 15(1 + 3\mathrm{g})\chi_{[2]} - 45(\mathrm{g} - 3)\chi_{[1,1]} - (15\mathrm{g} - 1)\frac{1}{2}\Big)$$
$$+ n_1^3 \cdot \big(360\chi_{[3]} + 375\chi_{[2,1]} + 240\chi_{[1,1,1]} - 20(-1 + 3\mathrm{g})\chi_{[1]}\big)$$
$$+ n_1^4 \cdot \big(270\chi_{[2]} + 240\chi_{[1,1]} - 5(-1 + 3\mathrm{g})\big)$$
$$+ 108\chi_{[1]} \cdot n_1^5 + 18 \cdot n_1^6\Big)\frac{1}{180}\,, \tag{45}$$

$$u_{6,8} = \big(4\big(\chi_{[6]} - \chi_{[5,1]} + \chi_{[4,1,1]} - \chi_{[3,1,1,1]} + \chi_{[2,1,1,1,1]} - \chi_{[1,1,1,1,1,1]}\big)$$
$$- 24\chi_{[2]} + 24\chi_{[1,1]} + 20\mathrm{g}$$
$$+ n_1 \cdot \big(24\big(\chi_{[5]} - \chi_{[4,1]} + \chi_{[3,1,1]} - \chi_{[2,1,1,1]} + \chi_{[1,1,1,1,1]}\big)$$
$$+ 240\chi_{[2,1]} - 48\chi_{[1]}\big) + n_1^2 \cdot \big(60\chi_{[4]} + 60\chi_{[3,1]} + 240\chi_{[2,2]}$$
$$+ 300\chi_{[2,1,1]} - 180\chi_{[1,1,1,1]} + 240\chi_{[1,1]} - 24\big)$$
$$+ n_1^3 \cdot \big(80\chi_{[3]} + 40\chi_{[2,1]} + 80\chi_{[1,1,1]}\big)$$
$$+ n_1^4 \cdot \big(60\chi_{[2]} + 60\chi_{[1,1]}\big) + 24\chi_{[1]} \cdot n_1^5 + 4 \cdot n_1^6\big)\frac{1}{2880}\,, \tag{46}$$

$$u_{6,9} = \big(3\big(\chi_{[6]} - \chi_{[5,1]} + \chi_{[4,1,1]} - \chi_{[3,1,1,1]}$$
$$+ \chi_{[2,1,1,1,1]} - \chi_{[1,1,1,1,1,1]}\big) - 2\chi_{[2]} + 2\chi_{[1,1]} - \mathrm{g}$$
$$+ n_1 \cdot \big(18\big(\chi_{[5]} - \chi_{[4,1]} + \chi_{[3,1,1]}$$
$$- \chi_{[2,1,1,1]} + \chi_{[1,1,1,1,1]}\big) + 20\chi_{[2,1]} - 4\chi_{[1]}\big)$$
$$+ n_1^2 \cdot \big(45\chi_{[4]} + 5\chi_{[3,1]} - 20\chi_{[2,2]}$$
$$+ 25\chi_{[2,1,1]} + 25\chi_{[1,1,1,1]} + 20\chi_{[1,1]} - 2\big)$$
$$+ n_1^3 \cdot \big(60\chi_{[3]} + 70\chi_{[2,1]} + 60\chi_{[1,1,1]}\big)$$
$$+ n_1^4 \cdot \big(45\chi_{[2]} + 45\chi_{[1,1]}\big) + 18\chi_{[1]} \cdot n_1^5 + 3 \cdot n_1^6\big)\frac{1}{240}\,. \tag{47}$$

In this case, we also have the following:

(1) coefficients of leading terms do not depend on $g$, i.e. they are constants;
(2) coefficients in $u_{5,4}$ are constants;
(3) three following combinations have constant coefficients:

$$120u_{6,9} - u_{2,1}\,, \quad 12u_{4,3} - u_{2,1}\,, \quad 72u_{6,8} + u_{2,1}\,. \tag{48}$$

4.2.3. *Mixed*

$$w_{2,1} = \big(\chi_{[2]} - \chi_{[1,1]} - \mathrm{g} + n_1\big(-2\chi_{[1]}\big)\big)\frac{1}{2}\,, \tag{49}$$

$$w_{3,1} = \big(2\big(\chi_{[3]} - \chi_{[2,1]} + \chi_{[1,1,1]}\big) - 2\chi_{[1]} + n_1 3(-2\chi_{[2]} + \mathrm{g}) + n_1^2 3\chi_{[1]}\big)\frac{1}{6}\,, \tag{50}$$

$$\begin{aligned} w_{4,2} = \big(&7\big(\chi_{[4]} - \chi_{[3,1]} + \chi_{[2,1,1]} - \chi_{[1,1,1,1]}\big) - 8\big(\chi_{[2]} - \chi_{[1,1]}\big) + \mathrm{g} \\ &+ n_1 \cdot \big(-28\chi_{[3]} + 4\chi_{[2,1]} - 4\chi_{[1,1,1]} + 4(4+3\mathrm{g})\chi_{[1]}\big) \\ &+ n_1^2 \cdot \big(30\chi_{[2]} + 18\chi_{[1,1]} - 6\mathrm{g}\big) - n_1^3 \cdot 4\chi_{[1]}\big)\frac{1}{24}\,, \end{aligned} \tag{51}$$

$$\begin{aligned} w_{4,3} = \big(&\chi_{[4]} - \chi_{[3,1]} + \chi_{[2,1,1]} - \chi_{[1,1,1,1]} - \mathrm{g} \\ &+ n_1 \cdot \big(-4\big(\chi_{[3]} - \chi_{[2,1]} + \chi_{[1,1,1]}\big) \\ &+ 8\chi_{[1]}\big) + n_1^2 \cdot \big(6\chi_{[2]} + 6\chi_{[1,1]}\big)\big)\frac{1}{24}\,, \end{aligned} \tag{52}$$

$$\begin{aligned} w_{5,2} = \big(&102\big(\chi_{[5]} - \chi_{[4,1]} + \chi_{[3,1,1]} - \chi_{[2,1,1,1]} + \chi_{[1,1,1,1,1]}\big) \\ &- 140\big(\chi_{[3]} - \chi_{[2,1]} + \chi_{[1,1,1]}\big) \\ &+ 38\chi_{[1]} + n_1 \cdot \big(-510\chi_{[4]} + 90\chi_{[3,1]} + 60\chi_{[2,2]} - 90\chi_{[2,1,1]} \\ &+ 30\chi_{[1,1,1,1]} + 60(8+3\mathrm{g})\chi_{[2]} - 15\mathrm{g}(3\mathrm{g}+4)\big) \\ &+ n_1^2 \cdot \big(810\chi_{[3]} + 540\chi_{[2,1]} + 90\chi_{[1,1,1]} - 180(1+2\mathrm{g})\chi_{[1]}\big) \\ &+ n_1^3 \cdot \big(-360\chi_{[2]} - 300\chi_{[1,1]} + 30\mathrm{g}\big) + n_1^4 \cdot 15\chi_{[1]}\big)\frac{1}{360}\,, \end{aligned} \tag{53}$$

$$\begin{aligned} w_{5,3} = \big(&9\big(\chi_{[5]} - \chi_{[4,1]} + \chi_{[3,1,1]} - \chi_{[2,1,1,1]} + \chi_{[1,1,1,1,1]}\big) \\ &- 10\big(\chi_{[3]} - \chi_{[2,1]} + \chi_{[1,1,1]}\big) + \chi_{[1]} \\ &+ n_1 \cdot \big(-45\chi_{[4]} + 15\chi_{[3,1]} + 30\chi_{[2,2]} \\ &- 15\chi_{[2,1,1]} - 15\chi_{[1,1,1,1]} + 60\chi_{[2]} - 15\mathrm{g}\big) \\ &+ n_1^2 \cdot \big(75\chi_{[3]} + 60\chi_{[2,1]} - 15\chi_{[1,1,1]} - 45\mathrm{g}\chi_{[1]}\big) \\ &+ n_1^3 \cdot \big(-30\chi_{[2]} - 30\chi_{[1,1]}\big)\big)\frac{1}{180}\,, \end{aligned} \tag{54}$$

$$\begin{aligned} w_{5,4} = {} & \big(\chi_{[5]} - \chi_{[4,1]} + \chi_{[3,1,1]} - \chi_{[2,1,1,1]} + \chi_{[1,1,1,1,1]} - \chi_{[1]} \\ & + n_1 \cdot \big(-5\big(\chi_{[4]} - \chi_{[3,1]} + \chi_{[2,1,1]} - \chi_{[1,1,1,1]}\big) + 10\chi_{[2]}\big) \\ & + n_1^2 \cdot \big(10\chi_{[3]} + 5\chi_{[2,1]} + 10\chi_{[1,1,1]} - 10\chi_{[1]}\big) \\ & + n_1^3 \cdot \big(-5\chi_{[2]} - 5\chi_{[1,1]}\big)\big)\,\frac{1}{30}\,, \end{aligned} \tag{55}$$

$$\begin{aligned} w_{6,5} = {} & \big(203\big(\chi_{[6]} - \chi_{[5,1]} + \chi_{[4,1,1]} - \chi_{[3,1^3]} + \chi_{[2,1^4]} - \chi_{[1^6]}\big) \\ & - 340\big(\chi_{[4]} - \chi_{[3,1]} + \chi_{[2,1,1]} - \chi_{[1^4]}\big) + 140\big(\chi_{[2]} - \chi_{[1,1]}\big) - 3\mathrm{g} \\ & + n_1 \cdot \big(-1218\chi_{[5]} + 198\chi_{[4,1]} + 180\chi_{[3,2]} - 198\chi_{[3,1,1]} \\ & - 60\chi_{[2,2,1]} + 78\chi_{[2,1,1,1]} - 18\chi_{[1,1,1,1,1]} + +20(74 + 21\mathrm{g})\chi_{[3]} \\ & - 20(2 + 3\mathrm{g})\chi_{[2,1]} + 20(2 + 3\mathrm{g})\chi_{[1,1,1]} - 2(45\mathrm{g}^2 + 180\mathrm{g} + 86)\chi_{[1]}\big) \\ & + n_1^2 \cdot \big(2535\chi_{[4]} + 1725\chi_{[3,1]} + 420\chi_{[2,2]} + 255\chi_{[2,1,1]} + 45\chi_{[1^4]} \\ & - 30(44 + 45\mathrm{g})\chi_{[2]} - 30(20 + 21\mathrm{g})\chi_{[1,1]} + 15\mathrm{g}(12\mathrm{g} + 7)\big) \\ & + n_1^3 \cdot \big(-1960\chi_{[3]} - 2300\chi_{[2,1]} - 640\chi_{[1,1,1]} + 20(8 + 33\mathrm{g})\chi_{[1]}\big) \\ & + n_1^4 \cdot \big(405\chi_{[2]} + 375\chi_{[1,1]} - 15\mathrm{g}\big) - 6\chi_{[1]} \cdot n_1^5\big)\,\frac{1}{720}\,, \end{aligned} \tag{56}$$

$$\begin{aligned} w_{6,6} = {} & \Bigg(5\big(\chi_{[6]} - \chi_{[5,1]} + \chi_{[4,1,1]} - \chi_{[3,1^3]} + \chi_{[2,1^4]} - \chi_{[1^6]}\big) \\ & - 30\big(\chi_{[4]} - \chi_{[3,1]} + \chi_{[2,1,1]} - \chi_{[1^4]}\big) + 19\big(\chi_{[2]} - \chi_{[1,1]}\big) + 6\mathrm{g} \\ & + n_1 \cdot \big(-30\chi_{[5]} - 60\chi_{[4,1]} + 30\chi_{[3,2]} + 60\chi_{[3,1,1]} + 30\chi_{[2,2,1]} \\ & - 120\chi_{[2,1,1,1]} + 90\chi_{[1,1,1,1,1]} + 10(3\mathrm{g} - 1)\chi_{[3]} \\ & - 10(-4 + 3\mathrm{g})\chi_{[2,1]} + 10(-13 + 3\mathrm{g})\chi_{[1,1,1]} - 6(-2 + 5\mathrm{g})\chi_{[1]}\big) \\ & + n_1^2 \cdot \Bigg(30\chi_{[4]} + 135\chi_{[3,1]} + 60\chi_{[2,2]} - 180\chi_{[2,1,1]} - 15\chi_{[1^4]} \\ & - 15(9\mathrm{g} - 16)\chi_{[2]} - 45(\mathrm{g} - 2)\chi_{[1,1]} + \frac{15\mathrm{g}}{2}(3\mathrm{g} - 1)\Bigg) \\ & + n_1^3 \cdot \big(15\chi_{[3]} + 30\chi_{[2,1]} + 135\chi_{[1,1,1]} + 20(-4 + 3\mathrm{g})\chi_{[1]}\big)\Bigg)\,\frac{1}{360}\,, \end{aligned} \tag{57}$$

$$\begin{aligned} w_{6,7} = {} & \big(\big(\chi_{[6]} - \chi_{[5,1]} + \chi_{[4,1,1]} - \chi_{[3,1^3]} + \chi_{[2,1^4]} - \chi_{[1^6]}\big) \\ & - 20\big(\chi_{[4]} - \chi_{[3,1]} + \chi_{[2,1,1]} - \chi_{[1^4]}\big) - 14\big(\chi_{[2]} - \chi_{[1,1]}\big) - 2\mathrm{g} \\ & + n_1 \cdot \big(-216\chi_{[5]} + 156\chi_{[4,1]} + 60\chi_{[3,2]} - 156\chi_{[3,1,1]} - 60\chi_{[2,2,1]} \\ & + 156\chi_{[2,1^3]} - 96\chi_{[1^5]} + 380\chi_{[3]} - 80\chi_{[2,1]} + 140\chi_{[1^3]} - 12(4 + 5\mathrm{g})\chi_{[1]}\big) \end{aligned}$$

$$
\begin{aligned}
&+ n_1^2 \cdot \big(510\chi_{[4]} + 180\chi_{[3,1]} - 60\chi_{[2,2]} + 270\chi_{[2,1,1]} + 60\chi_{[1^4]} \\
&- 30(22+3\mathrm{g})\chi_{[2]} - 30(10+3\mathrm{g})\chi_{[1,1]} + 45\mathrm{g}\big) \\
&+ n_1^3 \cdot \big(-450\chi_{[3]} - 600\chi_{[2,1]} - 330\chi_{[1,1,1]} + 60(2+\mathrm{g})\chi_{[1]}\big) \\
&+ 75 n_1^4 \cdot \big(\chi_{[2]} + \chi_{[1,1]}\big)\big)\frac{1}{360}\,,
\end{aligned}
\tag{58}
$$

$$
\begin{aligned}
w_{6,8} = &\big(\chi_{[6]} - \chi_{[5,1]} + \chi_{[4,1,1]} - \chi_{[3,1,1,1]} + \chi_{[2,1,1,1,1]} \\
&- \chi_{[1,1,1,1,1,1]} - 6\big(\chi_{[2]} + \chi_{[1,1]}\big) + 5\mathrm{g} \\
&+ n_1 \cdot \big(-6\chi_{[5]} + 6\chi_{[4,1]} - 6\chi_{[3,1,1]} + 6\chi_{[2,1,1,1]} \\
&- 6\chi_{[1,1,1,1,1]} + 20\chi_{[3]} + 40\chi_{[2,1]} + 20\chi_{[1,1,1]} - 4\chi_{[1]}\big) \\
&+ n_1^2 \cdot \big(15\chi_{[4]} + 15\chi_{[3,1]} + 60\chi_{[2,2]} \\
&+ 75\chi_{[2,1,1]} - 45\chi_{[1,1,1,1]} + 60\chi_{[1,1]}\big) \\
&+ n_1^3 \cdot \big(-10\chi_{[3]} + 40\chi_{[2,1]} - 10\chi_{[1,1,1]} + 40\chi_{[1]}\big) \\
&+ 15 n_1^4 \cdot \big(\chi_{[2]} + \chi_{[1,1]}\big)\big)\frac{1}{720}\,,
\end{aligned}
\tag{59}
$$

$$
\begin{aligned}
w_{6,9} = &\big(9\big(\chi_{[6]} - \chi_{[5,1]} + \chi_{[4,1,1]} - \chi_{[3,1,1,1]} + \chi_{[2,1,1,1,1]} - \chi_{[1,1,1,1,1,1]}\big) \\
&- 6\big(\chi_{[2]} + \chi_{[1,1]}\big) - 3\mathrm{g} + n_1 \cdot \big(-54\chi_{[5]} + 54\chi_{[4,1]} \\
&- 54\chi_{[3,1,1]} + 54\chi_{[2,1,1,1]} - 54\chi_{[1,1,1,1,1]} \\
&+ 100\chi_{[3]} - 40\chi_{[2,1]} + 100\chi_{[1,1,1]} - 52\chi_{[1]}\big) \\
&+ n_1^2 \cdot \big(135\chi_{[4]} + 15\chi_{[3,1]} - 60\chi_{[2,2]} + 75\chi_{[2,1,1]} \\
&+ 75\chi_{[1,1,1,1]} - 240\chi_{[2]} - 180\chi_{[1,1]}\big) \\
&+ n_1^3 \cdot \big(-130\chi_{[3]} - 200\chi_{[2,1]} - 130\chi_{[1,1,1]} + 40\chi_{[1]}\big) \\
&+ 15 n_1^4 \cdot \big(\chi_{[2]} + \chi_{[1,1]}\big)\big)\frac{1}{720}\,.
\end{aligned}
\tag{60}
$$

In this case, we also have the following:

(1) Coefficients of leading terms do not depend on $g$, i.e. they are constants;
(2) Coefficients in $w_{5,4}$ are constants;
(3) Three following combinations have constant coefficients:

$$120 w_{6,9} - w_{2,1}\,, \quad 12 w_{4,3} - w_{2,1}\,, \quad 72 w_{6,8} + w_{2,1}\,. \tag{61}$$

Thus, we see that these three observations are valid for all pretzel subfamilies, i.e. they are universal for any pretzel knot. It is very promising, probably, it helps to find a distinguished basis in the space of chord diagrams, because the current one (trivalent diagrams) is accidental.

## 5. Properties of the Vassiliev Invariants

### 5.1. *Distinguishing knots*

How many of the Vassiliev invariants are needed to distinguish pretzel knots? In the case of torus knots, the answer was found in Ref. 21. The Vassiliev invariants of the second and third orders are enough. In the case of pretzel knots, the answer is unknown at the present moment. We definitely know that only $v_{2,1}^{\mathcal{K}}$ is not enough. For example, knots $(3,3,3)$ and $(-3,5,21)$ have same second Vassiliev invariants but different HOMFLY polynomials.

### 5.2. *Topological information*

Which Vassiliev invariants contain topological information? In other words, we are looking for relations among them additional to (10). In the case of torus knots, there are only one independent Vassiliev invariant at each order up to order 6.[21] In the case of pretzel knots, we found the only relation at order 6 only for antiparallel subfamily:

$$\begin{aligned} &-103v_{2,1} + 240v_{4,3} + 1080v_{6,6} - 180v_{6,7} + 630v_{6,8} + 4770v_{6,9} \\ &\quad - 60v_{2,1}^2 + 90v_{2,1}^3 - 180v_{3,1}^2 - 180v_{2,1}v_{4,2} - 1080v_{2,1}v_{4,3} = 0\,. \end{aligned} \tag{62}$$

There are no more universal relations. We can say that antiparallel pretzel subfamily contains less topological information than two others. This feature is rather surprising and deserves further studies.

### 5.3. *Integer-valued*

*These results are valid for all families of pretzel knots.*

Let us rescale Vassiliev invariants by normalization on the trefoil

$$\tilde{v}_{i,j}^{\mathcal{K}} = \frac{v_{i,j}^{\mathcal{K}}}{v_{i,j}^{3_1}} \tag{63}$$

and multiply them by the following factors

$$\begin{gathered} \tilde{v}_{2,1},\ \tilde{v}_{3,1},\ 31\tilde{v}_{4,2},\ 5\tilde{v}_{4,3},\ 11\tilde{v}_{5,2},\ \tilde{v}_{5,3},\ \tilde{v}_{5,4}, \\ 5071\tilde{v}_{6,5},\ 29\tilde{v}_{6,6},\ 1531\tilde{v}_{6,7},\ 17\tilde{v}_{6,8},\ 271\tilde{v}_{6,9}\,, \end{gathered} \tag{64}$$

then so defined Vassiliev invariants take only integer values for all knots. For orders $i = 2, 3, 4$, we can prove it by the straightforward enumerations, for $i = 5, 6$, we have a lot of numerical results

$$\begin{gathered} v_{2,1}^{3_1} = 4\,, \quad v_{3,1}^{3_1} = -8\,, \quad v_{4,2}^{3_1} = \frac{62}{3}\,, \quad v_{4,3}^{3_1} = \frac{10}{3}\,, \\ v_{5,2}^{3_1} = -\frac{176}{3}\,, \quad v_{5,3}^{3_1} = -\frac{32}{3}\,, \quad v_{5,4}^{3_1} = -8\,, \end{gathered} \tag{65}$$

$$v_{6,5}^{3_1} = \frac{5071}{30}\,, \quad v_{6,6}^{3_1} = \frac{58}{15}\,, \quad v_{6,7}^{3_1} = \frac{3062}{45}\,, \quad v_{6,8}^{3_1} = \frac{17}{18}\,, \quad v_{6,9}^{3_1} = \frac{271}{30}\,. \tag{66}$$

## Acknowledgments

Our work is partly supported by grants NSh-1500.2014.2, by RFBR grants 13-02-00457, by the joint grants 15-52-50034-YaF, 15-51-52031-NSC-a, by 14-01-92691-Ind-a, 14-02-31446-Mol-a, 15-31-20832-Mol-a-ved, mol-a-dk 16-31-60082 and by MK-8769.2016.1. Also we are partly supported by the Quantum Topology Lab of Chelyabinsk State University (Russian Federation Government Grant 14.Z50.31.0020).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