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Monoid and group of pseudo braids

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ABSTRACT

In this paper, we introduce pseudo braid monoids and show that, such a monoid is isomorphic to a singular braid monoid. We also prove an analogue of Markov's Theorem for pseudo braids.

Keywords: Pseudo knots; pseudo braids; singular knots; Markov's Theorem.

Mathematics Subject Classification 2010: 20F18, 20D15, 20E05

1. Introduction

In 2010, Hanaki [8] introduced the notion of pseudo diagrams of knots, links and spatial graphs. A pseudo knot diagram is similar to the projection of a knot in $\mathbb{R}^2$, but besides over-crossings and under-crossings, we allow unspecified crossings (which could be either over or under crossings). A double point with over/under information and a double point without over/under information are called a *crossing* and a *pre-crossing*, respectively. The study of DNA knots serves a motivation for this research. Namely, we cannot determine over/under information at some of the crossings in photos of DNA knots. Even though, DNA knots barely become visual objects by examining the protein-coated one by electron microscopes, there are still cases in which it is hard to confirm the over/under information of some crossings.

If we know the (non-)triviality of a knot without checking over/under information for each crossing, then it provides a reasonable way to detect the (non-)triviality of DNA knots.

Pseudo knots were subsequently introduced in [9] and defined as equivalence classes of pseudo diagrams under an appropriate choice of Reidemeister moves. In order to classify pseudo knots, the authors of [9] introduced the concept of a weighted resolution set, an invariant of pseudo knots, and computed the weighted resolution set for several pseudo knots families and discussed extensions of crossing numbers, homotopy, and chirality for pseudo knots. The aim of this paper is to address the following question formulated in their paper.

Question 1 ([9, Question 3]). What is an appropriate definition of pseudo braids? In particular, when are two pseudo braids equivalent? Furthermore, in classical braid theory, there are Markov moves that characterize, when two braids have equivalent closure. Is there an analog for pseudo braids?

Pseudo knots are closely related to singular knots. There is a map f from the set of singular knot diagrams to the set of pseudo knot diagrams, where singular crossings are replaced by pre-crossings. The map f is bijective, because for every pseudo knot diagram, we can find a singular knot diagram, where the singular crossings replace the pre-crossings and vice versa. Modulo equivalence relations defining pseudo or singular knots, the map f induces an onto map, denoted F, from singular knots/links to pseudo knots/links because the image of two isotopic singular knot diagrams are also isotopic pseudo knot diagrams with exactly the same sequence of Reidemeister moves (where the singular crossings are replaced by pre-crossings):

$$F : S\mathcal{L} \to P\mathcal{L}. \tag{1.1}$$

Respectively, $S\mathcal{L}, P\mathcal{L}$ denote the set of singular links and that of pseudo links. But note that this map F is not one-to-one. The key idea for solving Question 1 is to present a pseudo knot K using a singular knot in the pre-image $F^{-1}(K)$ and using existing formulations of singular knots to deduce our main theorem (Theorem 4.2) for pseudo knots.

2. Monoids of Pseudo Braids

To properly define the monoid of pseudo braids, we recall the definition of a braid and a singular monoid. The braid group B_n of n strands is generated by σ_i, σ_i^{-1} for $i = 1, \ldots, n-1$ subject to the relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i \quad |i-j| > 1, \tag{2.1}$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \quad 1 \le i \le n-2. \tag{2.2}$$

The generators σ_i, σ_i^{-1} and the relations (2.1) and (2.2) are presented by the following diagrams (see Fig. 1).

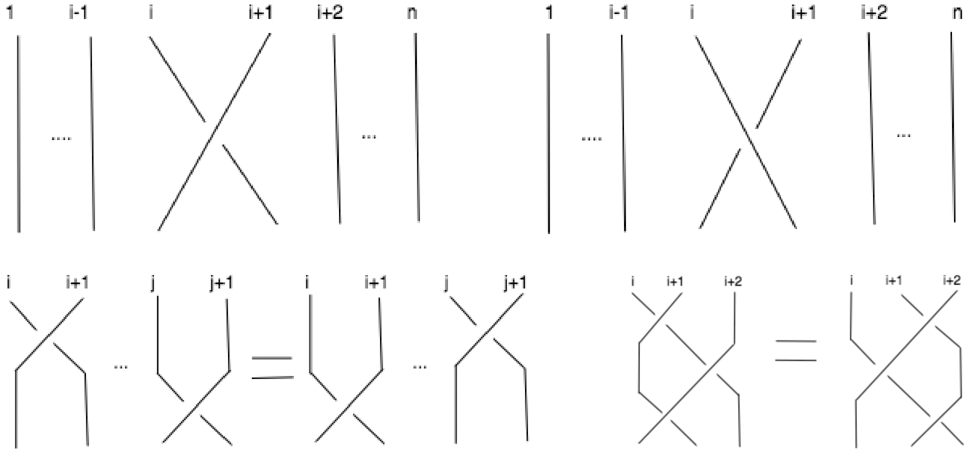

Fig. 1. σ_i, σ_i^{-1} and relations (2.1) and (2.2).

Definition 2.1 (Baez–Birman [1, 4]). The Baez–Birman monoid or the singular braid monoid SM_n is generated (as a monoid) by elements σ_i, σ_i^{-1}, τ_i, $i = 1, 2, \ldots, n-1$ subject to the following relations: the relations for the elements σ_i, σ_i^{-1} generating the braid group B_n; defining relations for the generators τ_i:

$$\tau_i \tau_j = \tau_j \tau_i, \quad |i - j| \geq 2 \tag{2.3}$$

and other mixed relations:

$$\tau_i \sigma_j^{\pm 1} = \sigma_j^{\pm 1} \tau_i, \quad |i - j| \geq 2, \tag{2.4}$$

$$\tau_i \sigma_i^{\pm 1} = \sigma_i^{\pm 1} \tau_i, \quad i = 1, 2, \ldots, n-1, \tag{2.5}$$

$$\sigma_i \sigma_{i+1} \tau_i = \tau_{i+1} \sigma_i \sigma_{i+1}, \quad i = 1, 2, \ldots, n-2, \tag{2.6}$$

$$\sigma_{i+1} \sigma_i \tau_{i+1} = \tau_i \sigma_{i+1} \sigma_i, \quad i = 1, 2, \ldots, n-2. \tag{2.7}$$

Fenn–Keyman–Rourke [6] proved that the singular braid monoid SM_n is embedded into the group SB_n, which is called *the singular braid group* and has the same defining relations as SM_n.

The generators τ_i, $i = 1, 2, \ldots, n-1$ have a geometric interpretation (see Fig. 2).

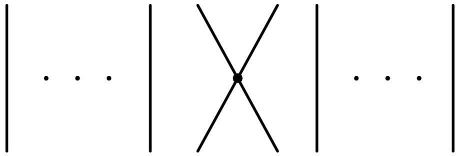

Fig. 2. The generator τ_i.

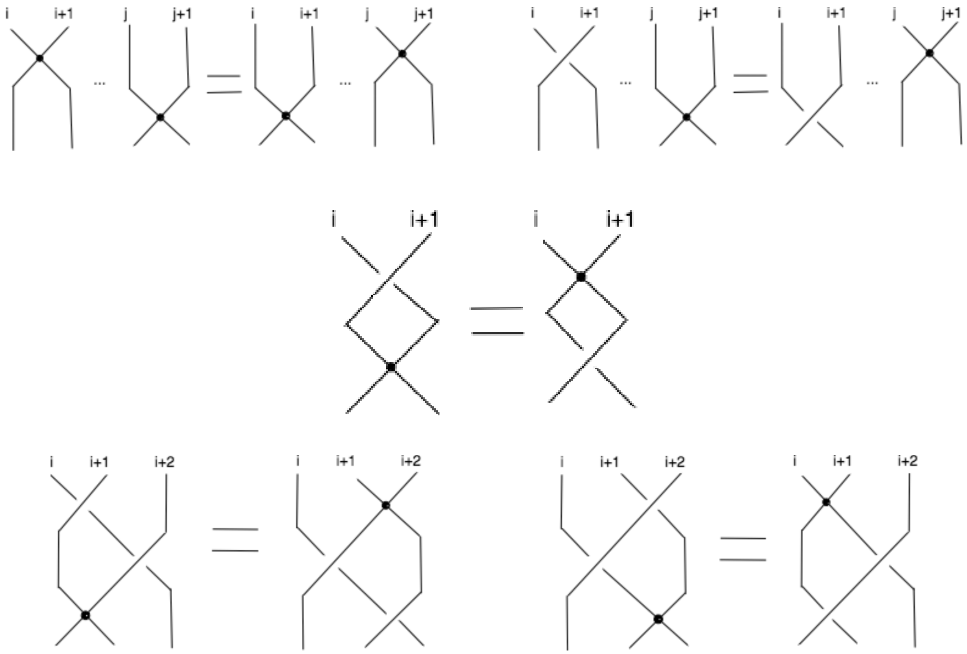

Fig. 3. Relations (2.3)–(2.7).

where the ith and $(i + 1)$th strings intersect. The topological interpretations of relations (2.3)–(2.7) are described in Fig. 3, respectively.

To define a monoid of pseudo braids PM_n, we take the generators $\sigma_1^{\pm 1}, \sigma_2^{\pm 1}, \ldots, \sigma_{n-1}^{\pm 1}$ of the braid group B_n and add the generators $p_1, p_2, \ldots, p_{n-1}$, which are similar to the generators $\tau_1, \tau_2, \ldots, \tau_{n-1}$, but they denote pre-crossings instead of singular crossings (see Fig. 4).

To find defining relations, we make use of pseudo-Reidemeister moves for the pseudo links introduced in [9] (see Fig. 5).

The Reidemeister moves $R2$ and $R3$ correspond to relations

$$\sigma_i \sigma_i^{-1} = \sigma_i^{-1} \sigma_i, \tag{2.8}$$

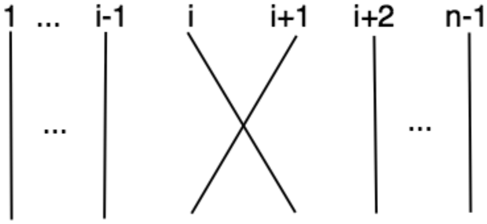

Fig. 4. The generator p_i.

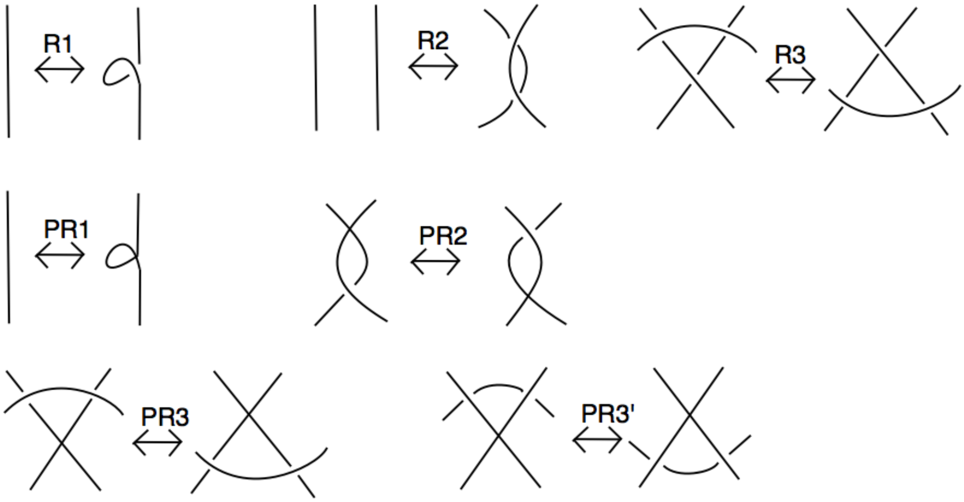

Fig. 5. Pseudo-Reidemeister moves.

$$\sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i. \tag{2.9}$$

Note that these are relations in B_n.

The move $PR2$ (see Fig. 5) corresponds to relations

$$p_i\sigma_i^{\pm 1} = \sigma_i^{\pm 1}p_i. \tag{2.10}$$

The moves $PR3$ and $PR3$' (see Figs. 6 and 7) correspond to relations

$$\sigma_i\sigma_{i+1}p_i = p_{i+1}\sigma_i\sigma_{i+1}, \tag{2.11}$$

$$\sigma_{i+1}\sigma_i p_{i+1} = p_i\sigma_{i+1}\sigma_i. \tag{2.12}$$

The first Reidemeister moves $R1$ and $PR1$ do not hold in PM_n. Hence, we can formulate:

Definition 2.2 (Monoid and group of pseudo braids). The *monoid of pseudo braids* PM_n is a monoid generated by σ_i, σ_i^{-1}, p_i, $i = 1, 2, \ldots, n-1$, where the elements $\sigma_i^{\pm 1}$ generate the braid group B_n and generators p_i satisfy the defining

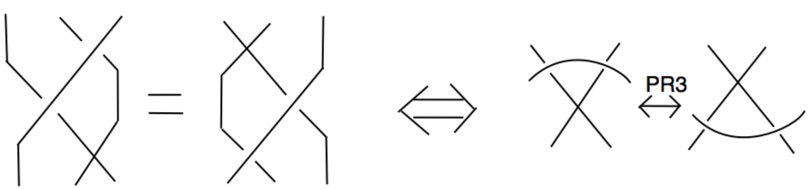

Fig. 6. The Reidemeister move $PR3$.

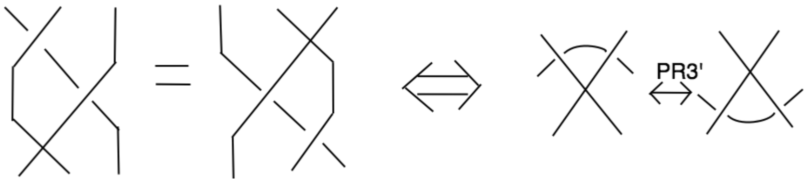

Fig. 7. The Reidemeister move $PR3$'.

relations

$$p_i p_j = p_j p_i, \quad |i-j| \geq 2 \tag{2.13}$$

and other mixed relations

$$p_i \sigma_j^{\pm 1} = \sigma_j^{\pm 1} p_i, \quad |i-j| \geq 2, \tag{2.14}$$

$$p_i \sigma_i^{\pm 1} = \sigma_i^{\pm 1} p_i, \quad i = 1, 2, \ldots, n-1, \tag{2.15}$$

$$\sigma_{i+1} \sigma_i p_{i+1} = p_i \sigma_{i+1} \sigma_i, \quad i = 1, 2, \ldots, n-2, \tag{2.16}$$

$$\sigma_i \sigma_{i+1} p_i = p_{i+1} \sigma_i \sigma_{i+1}, \quad i = 1, 2, \ldots, n-2. \tag{2.17}$$

The *group of pseudo braids* PB_n is a group generated by σ_i, p_i, $i = 1, 2, \ldots, n-1$ and defined by the same defining relations as PM_n.

Comparing the defining relations for SM_n and PM_n, we see that they are isomorphic and the isomorphism $SM_n \longrightarrow PM_n$ is defined by the rule $\sigma_i^{\pm} \mapsto \sigma_i^{\pm}$ and $\tau_i \mapsto p_i$ for all $i = 1, 2, \ldots, n-1$. On the other side, in [6], it was proved that SM_n is embedded into the the group SB_n, which is called the *singular braid group* and has the same defining relations as SM_n. Hence, we have:

Proposition 2.3. *The monoid of pseudo braids PM_n is isomorphic to the singular braid monoid SM_n and the group of pseudo braids PB_n is isomorphic to the group of singular braids SB_n for all $n \geq 2$.*

Remark 2.4. Comparing the Reidemeister moves for the singular links and for pseudo links, we see that the difference is the first Reidemeister move $PR1$ (see Fig. 5), which does not hold for singular links, but holds for pseudo links. Hence, the theory of pseudo links is the quotient of the theory of singular links by the first singular Reidemeister move.

Since the monoid of pseudo braids PM_n is isomorphic to the singular braid monoid SM_n, we can reformulate properties of SM_n for PM_n. There are some different ways to solve the word problem for the singular braid monoid and for the singular braid group. The singular braid monoid on two strings is isomorphic to $\mathbb{Z} \times \mathbb{Z}^+$, so the word problem in this case is trivial. In the general case a solution of the word problem in SM_n was done by Corran [5]. Vershinin [11] generalized

Garside's results and constructed the greedy normal form for SM_n. We do not know if it is possible to generalize this result to the group SB_n. Paris [10] proved that SM_n is a semi-direct product of a right-angled Artin monoid and B_n. This gives a solution of the word problem for SM_n and SB_n.

The desingularization map is the multiplicative homomorphism $\eta : SM_n \to \mathbb{Z}[B_n]$ defined by $\eta(\sigma_i^{\pm 1}) = \sigma_i^{\pm 1}$ and $\eta(\tau_i) = \sigma_i - \sigma_i^{-1}$ for $1 \leq i \leq n-1$. This homomorphism is one of the main ingredients of the definition of Vassiliev invariants for braids (see [4]). Paris [10] proved that the desingularization map η is an embedding of SM_n into the group algebra $\mathbb{C}[B_n]$. It provides an answer to the question of Birman [4]. Hence, the homomorphism η gives other solutions of the word problem in SM_n.

To solve the word problem in the groups SB_n and hence in PB_n, we need to define an inverse of τ_i. Consider the group algebra $\mathbb{C}[[B_n]]$ of formal power series. If an element $a \in \mathbb{C}[[B_n]]$ has the inverse $a^{-1} \in \mathbb{C}[[B_n]]$, then

$$(a - a^{-1})^{-1} = a(a^2 - 1)^{-1}.$$

Since

$$(a^2 - 1)^{-1} = -(1 + a^2 + a^4 + \cdots)$$

we have proved the following proposition.

Proposition 2.5. *There is a homomorphism* $\widetilde{\eta} : SB_n \to \mathbb{C}[[B_n]]$ *which is defined on the generators*

$$\tau_i \mapsto \sigma_i - \sigma_i^{-1}, \quad \sigma_i \mapsto \sigma_i$$

and is an extension of the homomorphism η. *Under this homomorphism, we have*

$$\tau_i^{-1} \mapsto (\sigma_i - \sigma_i^{-1})^{-1} = -\sigma_i(1 + \sigma_i^2 + \sigma_i^4 + \cdots).$$

Question 2. Is it true that the homomorphism $\widetilde{\eta} : SB_n \longrightarrow \mathbb{Z}[[B_n]]$ is an embedding?

If it is true, then we will have a solution of the word problem for the groups SB_n and PB_n.

3. Geometric Interpretation of Pseudo Braids and Alexander's Theorem

The geometric interpretation for the generators p_i, $i = 1, 2, \ldots, n-1$ is given by Fig. 4. Any two geometric pseudo braids for the same n can be composed into their product (see Fig. 8).

Every pseudo braid corresponds a pseudo knot or link by taking its closure (see Fig. 9).

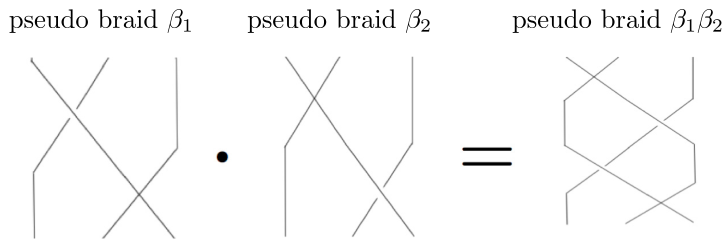

Fig. 8. Product of two pseudo braids β_1 and β_2.

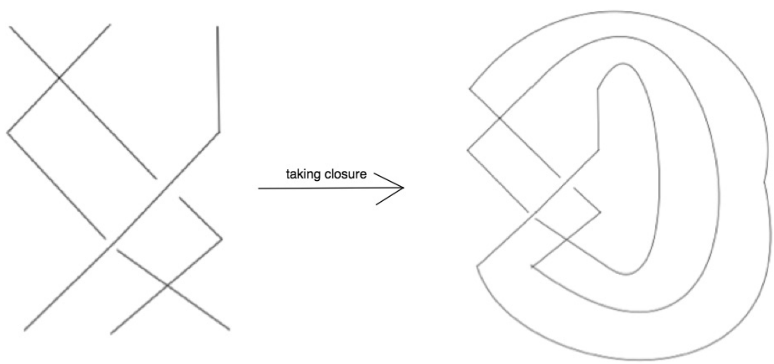

Fig. 9. A pseudo braid β and its closure $\widehat{\beta}$.

Denote the closure of a pseudo braid β by $\widehat{\beta}$. This closure operation gives rise to a map from the set of pseudo braids to the set of pseudo links

$$\widehat{\ }: PM \to P\mathcal{L},$$

where $PM = \bigcup_{n=2}^{\infty} PM_n$ is the inductive limit associated to the injective homomorphisms of pseudo braid monoids $PM_n \to PM_{n+1}, n \geq 2$ and $P\mathcal{L}$ is the set of pseudo links.

Birman [4] proved Alexander's Theorem for singular links. This result allows us to prove the following analogue of Alexander's Theorem for pseudo links.

Theorem 3.1. *Let L be a pseudo link. Then there exists a pseudo braid $\beta \in PM_n$ for some n, such that the closure $\widehat{\beta}$ is equivalent to L.*

Proof. Because there is a natural map F from singular links onto pseudo links (see (1.1)), let L' be a singular link in $F^{-1}(L)$. From [4], we know that there is a singular braid β', such that the closure of β' is the singular link L'. From Proposition 2.3, there is a unique pseudo braid β corresponding to β' under the monoid isomorphism and the closure of β is exactly the pseudo link L. □

4. Markov's Theorem

Consider the set of monoids of pseudo braids PM_n for $n = 2, 3, \ldots$ and let $PM = \bigcup_{n=2}^{\infty} PM_n$. Define the following Markov's moves on the set PM:

$M1$. If $\beta \in PM_n$ and $a \in B_n$, then

$$\beta \leftrightarrow a^{-1}\beta a.$$

$M2$. If $\beta = \beta_1\beta_2 \in PM_n$, then

$$\beta \leftrightarrow \beta_2\beta_1.$$

$M3$ If $\beta \in PM_n$, then

$$\beta \leftrightarrow \beta\sigma_n^{\pm 1} \in PM_{n+1}.$$

$M4$. If $\beta \in PM_n$, then

$$\beta \leftrightarrow \beta p_n \in PM_{n+1}.$$

The topological interpretations of Markov's moves $M1$–$M4$ are described as follows (see Figs. 10(a)–10(d)).

Remark 4.1. Note that moves $M1$ are special cases of $M2$, i.e. $a^{-1}\beta a = \beta aa^{-1} = \beta$. We emphasis this special case, because moves $M1$, $M3$ are used in Markov's theorem of classical links. For singular links, a more general type of moves $M2$ is needed in Markov's theorem. In our situation, moves of type $M4$ are added to the list in

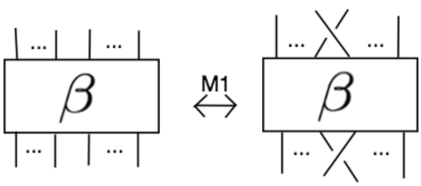

Fig. 10(a). $M1$ (conjugation by a braid group generator).

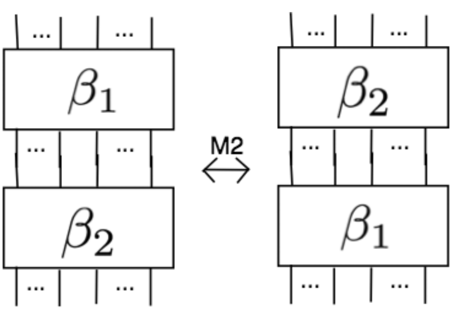

Fig. 10(b). $M2$ (permutation of pseudo braids in a product).

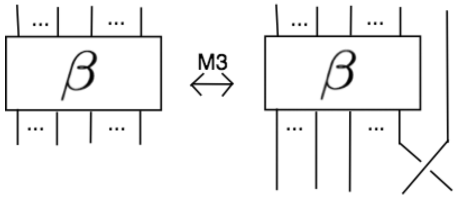

Fig. 10(c). $M3$ (Addition of an extra strand with the braid generator σ_n).

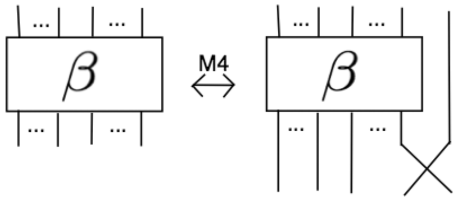

Fig. 10(d). $M4$ (Addition of an extra strand with the pre-crossing p_n).

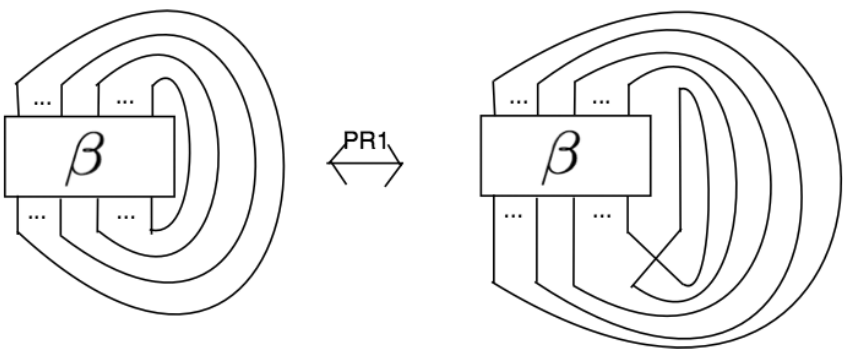

Fig. 11. Closure of two braids in an $M4$ move.

the Markov's theorem for pseudo links. The moves $M4$ are essential to characterize the first pseudo-Reidemeister move $PR1$. In particular, closures of pseudo links in Fig. 10(d) differ by the first pseudo-Reidemeister move (see Fig. 11).

Theorem 4.2. *Let L and L' be two pseudo link diagrams. Suppose that $L = \widehat{\beta}$ and $L' = \widehat{\beta'}$ for some pseudo braids $\beta \in PM_n$ and $\beta' \in PM_m$. Then the pseudo links L and L' are equivalent if and only if there is a finite sequence of Markov's moves ($M1$–$M4$), which transform β to β'.*

To prove this theorem, we will follow ideas of Birman [8, Chap. 2], who proved the Markov theorem for classical links and ideas of Gemein [7], who proved the

Markov theorem for singular links. For the singular case, the Markov's moves consist of the moves $M1$–$M3$. Birman's proof does not use Reidemeister moves but triangular moves. Gemein defines triangular moves, which can be applied to singular points. We modify his moves to our situation, so they can be applied to pre-crossings.

Proof [Proof of Theorem 4.2]. It is straight forward to check that, if β and β' are different by a Markov move of type $M1$–$M3$ or $M4$, their closures are equivalent pseudo link diagrams.

To prove the converse, it is sufficient to show that if L' and L differ by a generalized Reidemeister move of pseudo link diagrams, then β' can be transformed to β through Markov moves and pseudo braid isotopies. Let $l' = f^{-1}(L')$ and $l = f^{-1}(L)$, i.e. l', l are obtained from L', L by replacing pre-crossings by singular crossings. By the Alexander's Theorem for singular links, there exist singular braids α', α such that $\widehat{\alpha}' = l'$ and $\widehat{\alpha} = l$. By Proposition 2.3, we may assume that α, α' are obtained from β, β' by replacing pre-crossings by singular crossings.

There are two cases to consider at this stage:

(a) If L, L' differ by a pseudo-Reidemeister move induced by a singular Reidemeister move, i.e. not a type $PR1$ move, then l, l' differ by a singular Reidemeister move. Then by Gemein's Markov theorem for singular links [7], the singular braids α' and α are different by singular braid isotopies and the singular version of Markov moves $M1$–$M3$. Then obviously, β and β' differ by pseudo braid isotopies and Markov moves $M1$–$M3$.

(b) If L, L' differ by a first pseudo-Reidemeister move $PR1$, then l, l' are no longer isotopic. Note that, if a link diagram is placed in $\mathbb{R}^2$ differently (for example, differ by a rotation), then their corresponding pseudo braids are different by pseudo braid isotopies and Markov moves $M1$–$M3$ applying similar argument in Case (a) using Gemein's theorem. Therefore L, L' in $\mathbb{R}^2$ can be placed in such a way that all other parts of the two diagrams are identical except for the local pieces (1) and (2) in Fig. 12.

Without loss of generality, assume that L, L' consist of pieces of segments so that, when we orient the links by assigning an arrow to each piece, the arrow is either up or down, i.e. no horizontal segments. After arrows are assigned to each piece of the links, compared to L, there is one extra piece of up-arrow in L' (see Figs. 12 (1) and (2)). To turn L (respectively L') into a pseudo braid β (respectively β'), where $L = \widehat{\beta}$ (respectively $L' = \widehat{\beta}'$), we need to

- fix the down-arrows, and
- replace all the up-arrows by a pair of braids with down-arrows, where the new pair of braids has over-crossings to other part of the diagram.

For example, in Fig. 12, the up-arrow in (1) is replaced by a pair of braids pointing down in (3). To obtain β and β', we execute the above procedure first at the corresponding identical up-arrow pieces of L and L', and lastly at the

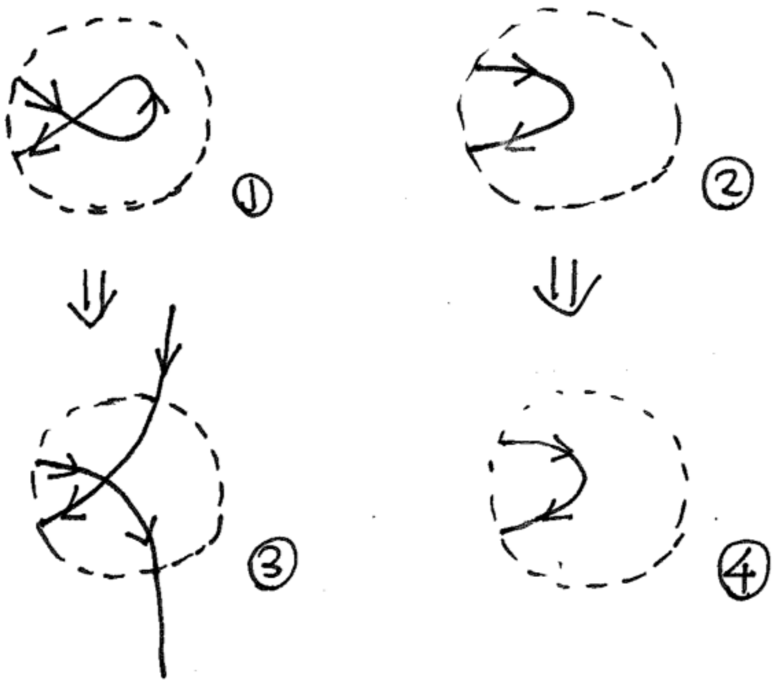

Fig. 12. Braiding of the pseudo-Reidemeister move $PR1$.

only different part of L, L', i.e. the local pieces (1) and (2) in Fig. 12. In this last step, (1) and (2) are turned into (3) and (4) in Fig. 12, respectively. Therefore, the two pseudo braids β and β' differ by a Markov move of type $M4$.

Because diagrams for the same pseudo link differ by a finite sequence of moves in either case (a) or (b), their corresponding braids are different by a finite sequence of Markov moves $M1$–$M4$. The theorem is then proved. □

In summary, pseudo links $P\mathcal{L}$ are characterized by closures of elements of PM. They are related to the singular links $S\mathcal{L}$ and singular braid manoids SM in the commutative diagram (4.1), in which the horizontal maps are closure of (singular or pseudo) braids and the right vertical map Q is the quotient by the first pseudo-Reidemeister moves $PR1$.

$$\begin{array}{ccc} SM & \longrightarrow & S\mathcal{L} \\ \cong \downarrow \text{Prop. 2.3} & & Q \downarrow \\ PM & \longrightarrow & P\mathcal{L}. \end{array} \tag{4.1}$$

The pseudo braid monoids PM admit an equivalence relations given by Markov moves $M1$–$M4$. Theorem 4.2 implies that the equivalence classes are in one-to-one correspondence with $P\mathcal{L}$. The theorem is related closely to Markov's theorem for singular links in the following commutative diagram, where the left vertical map q stands for taking quotient further by the Markov moves $M4$.

$$\begin{array}{ccc} SM/\{M1\text{–}M3\} & \xrightarrow{\cong} & S\mathcal{L} \\ \downarrow q & & Q \downarrow \\ PM/\{M1\text{–}M4\} & \xrightarrow[\cong]{\text{Th. 4.2}} & P\mathcal{L}. \end{array}$$

In conclusion, Theorem 4.2 transfers the classification problem of pseudo links to the problem of classifying equivalence classes of PM by the Markov moves $M1$–$M4$. In particular, a pseudo link l is trivial, if only only if a corresponding pseudo braid β, where $\widehat{\beta} = l$, is obtained as a sequence of Markov moves $M1$–$M4$ applied to a trivial braid.

Acknowledgments

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