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On discrete versions of two Accola's theorems about automorphism groups of Riemann surfaces

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Abstract In this paper we give a few discrete versions of Robert Accola's results on Riemann surfaces with automorphism groups admitting partitions. As a consequence, we establish a condition for γ-hyperelliptic involution on a graph to be unique. Also we construct an infinite family of graphs with more than one γ-hyperelliptic involution.

Keywords Riemann surface · Graph · Automorphism group · Hyperelliptic graph · Hyperelliptic involution · Harmonic map

Mathematics Subject Classification 05C10 · 57M12

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1 Introduction

Over the last decade, counterparts of many theorems from the classical theory of Riemann surfaces were derived in the discrete case [1–8]. In these theorems, the finite graphs play the role of Riemann surfaces, while the conformal automorphisms are replaced by harmonic ones.

In the present paper we give a few discrete versions of Robert Accola's results, [9,10], [11] Sect. 5.9, on Riemann surfaces with automorphism groups admitting partitions and a more general theorem (Theorems 1 and 2). The genus of a graph is defined as the rank of its homology group. A graph is said to be γ-hyperelliptic if it is a two fold branched covering of a genus γ graph. The corresponding covering involution is called γ-hyperelliptic. As a consequence of Theorem 1, we establish a condition for γ-hyperelliptic involution on a graph to be unique. Also we construct an infinite family of graphs with more than one γ-hyperelliptic involution.

2 Preliminaries

In this paper, a graph is a finite connected multigraph, possibly with loops. Denote by $V(X)$ and by $E(X)$ the set of vertices and the set of directed edges of graph X. We introduce two maps $\partial_0, \partial_1 : E(X) \to V(X)$ (endpoints) and a fixed point free involution $e \to \bar{e}$ of $E(X)$ (reversal of orientation) such that $\partial_i \bar{e} = \partial_{1-i} e$. We put

$$\mathrm{St}(a) = \mathrm{St}^X(a) = \partial_0^{-1}(a) = \{e \in E(X) \mid \partial_0 e = a\},$$

the *star* of a, and call $\deg(a) = |St(a)|$ the *degree* (or *valency*) of a. A *morphism* of graphs $\varphi : X \to Y$ carries vertices to vertices, edges to edges, and, for $e \in E(X)$, $\varphi(\partial_i e) = \partial_i \varphi(e)$ $(i = 0, 1)$ and $\varphi(\bar{e}) = \overline{\varphi(e)}$. For $a \in X$ we have the local map

$$\varphi_a : \mathrm{St}^X(a) \to \mathrm{St}^Y(\varphi(a)).$$

A map φ is *locally bijective* if φ_a is bijective for all $a \in X$. We call φ a *covering* if φ is surjective and locally bijective. A bijective morphism is called an *isomorphism*, and an isomorphism $\varphi : X \to X$ is called an *automorphism.*

Definition 1 A morphism $\varphi : X \to Y$ of graphs is said to be a *harmonic map* or *branched covering* if, for all $x \in V(X),\ y \in V(Y)$ such that $y = \varphi(x)$, the quantity

$$|e \in E(X) : x = \partial_0 e,\ \varphi(e) = e'|$$

is the same for all edges $e' \in E(Y)$ such that $y = \partial_0 e'$.

Note that the composition of two harmonic morphisms is again harmonic, and an arbitrary covering of graphs is a harmonic map.

Let $\varphi : X \to Y$ be harmonic and $x \in V(X)$. We define the *multiplicity* of φ at x by

$$m_\varphi(x) = |e \in E(X) : x = \partial_0 e,\ \varphi(e) = e'|$$

for any edge $e' \in E(Y)$ such that $\varphi(x) = \partial_0 e'$. By the definition of a harmonic morphism, $m_\varphi(x)$ is independent of the choice of e'.

Define the degree of a harmonic map $\varphi : X \to Y$ by the formula

$$\deg(\varphi) := |e \in E(X) : \varphi(e) = e'| \tag{1}$$

for any edge $e' \in E(Y)$. From the definition of a harmonic map of graphs and connectivity of the graphs, it follows that the right-hand side of (1) does not depend on the choice of e' and therefore $\deg(\varphi)$ is well defined.

Let $G < \mathrm{Aut}(X)$ be a group of automorphisms of a graph X. An edge $e \in E(X)$ is called *invertible* if there is $h \in G$ such that $h(e) = \bar{e}$. We say that the group G acts *harmonically* on a graph X if it acts freely on the set of directed edges of X. If G acts harmonically and without invertible edges, we say that G acts *purely harmonically* on X.

Let G act purely harmonically on X. Define the quotient graph X/G so that its vertices and edges are G-orbits of the vertices and edges of X. Note that if the endpoints of an edge $e \in E(X)$ lie in the same G-orbit then the G-orbit of e is a loop in the quotient graph X/G. In this case, the canonical projection $X \to X/G$ is a harmonic map.

The *genus* of a graph X is defined as the rank of the first homology group of X (that is, the Betti number or cyclomatic number of the graph). Let X, Y be graphs of genera g and γ respectively, and $\varphi : X \to Y$ be a harmonic map. By the same arguments as in [2], for the graph morphism under consideration we get an analogue of the Riemann–Hurwitz relation:

$$g - 1 = \deg(\varphi)(\gamma - 1) + \sum_{a \in V(X)} (m_\varphi(a) - 1),$$

where $m_\varphi(a)$ is the multiplicity of map φ at vertex a.

Let X be a graph of genus g, and $G < \mathrm{Aut}(X)$ act purely harmonically on X. Denote by γ the genus of X/G. Then the degree of $\varphi : X \to X/G$ is equal to $|G|$. If G^x denotes the stabiliser of $x \in V(X)$ in G, then $|G^x| = m_\varphi(x)$. Thus, the Riemann–Hurwitz formula for the canonical projection $\varphi : X \to X/G$ has the following form

$$g - 1 = |G|(\gamma - 1) + \sum_{x \in V(X)} (|G^x| - 1). \tag{2}$$

More general statement of the Riemann–Hurwitz formula for the groups acting on a graph with fixed and invertible edges can be found in [7].

3 Main results

A finite group G_0 is said to admit a *partition* $\{G_1, \dots, G_s\}$ if it can be expressed as a set-theoretic union of subgroups $G_1, \dots, G_s$ of G, with pairwise trivial intersections. The formula in the following theorem is of interest because the multiplicities of the canonical projections $X \to X/G_i$ do not occur.

Theorem 1 *Let X be a graph of genus g. Suppose $G_0 < \mathrm{Aut}(X)$ acts purely harmonically on X and admits a partition $\{G_1, \dots, G_s\}$. Let $n_i = |G_i|$, $g_i = g(X/G_i)$, where $i = 0, 1, \dots, s$. Then*

$$(s-1)g + n_0 g_0 = \sum_{i=1}^{s} n_i g_i .$$

Proof For the coverings $X \to X/G_0$ and $X \to X/G_i$ the Riemann–Hurwitz formula (2) gives

$$\begin{aligned} g - 1 &= n_0(g_0 - 1) + r_0, \\ g - 1 &= n_i(g_i - 1) + r_i, \end{aligned} \tag{3}$$

where r_0 and r_i are expressed in terms of the stabilisers G_0^v and G_i^v respectively

$$\begin{aligned} r_0 &= \sum_{v \in V(X)} (|G_0^v| - 1), \\ r_i &= \sum_{v \in V(X)} (|G_i^v| - 1). \end{aligned} \tag{4}$$

Since G_0 admits a partition $\{G_1, \dots, G_s\}$, we have

$$|G_0| - 1 = \sum_{1 \le i \le s} (|G_i| - 1),$$

or

$$n_0 - 1 = \sum_{1 \le i \le s} (n_i - 1). \tag{5}$$

In a similar way, since the stabiliser G_0^v is a group with the partition $\{G_1^v, G_2^v, \dots, G_s^v\}$, we have

$$|G_0^v| - 1 = \sum_{1 \le i \le s} (|G_i^v| - 1).$$

Summing the latter equality through all $v \in V(X)$, from (4) we obtain

$$r_0 = \sum_{1 \le i \le s} r_i .$$

Substituting (3) for r_0 and r_i in the latter equality, we have

$$(g-1) - n_0(g_0-1) = \sum_{1\le i\le s} [(g-1) - n_i(g_i-1)]$$

or

$$-n_0 g_0 + (n_0-1) = (s-1)g + \sum_{1\le i\le s} [-n_i g_i + (n_i-1)].$$

Taking into consideration (5), we obtain the result.

As the first two applications of Theorem 1 we consider the dihedral group, and the affine transformation group on a finite field. Let $\mathbb{D}_n$ be the dihedral group of order $2n$. Then $\mathbb{D}_n$ is a group admitting a partition. Indeed, let $R \in \mathbb{D}_n$ generate the cyclic subgroup $\langle R\rangle$ of order n and let V be an element of order two not in $\langle R\rangle$. Then $\{\langle R\rangle, \langle V\rangle, \langle R\,V\rangle, \ldots, \langle R^{n-1}V\rangle\}$ is a partition of D_n. Let X be a finite graph. Denote by $g(X)$ the genus of X.

Application 1 *Let $\mathbb{D}_n$ be the dihedral group purely harmonically acting on a finite graph X. Then we have*

$$g(X) + 2g(X/\mathbb{D}_n) = g(X/\langle R\rangle) + g(X/\langle V\rangle) + g(X/\langle R\,V\rangle).$$

Proof By Theorem 1 we have

$$ng(X) + 2ng(X/\mathbb{D}_n) = ng(X/\langle R\rangle) + 2\sum_{i=0}^{n-1} g(X/\langle R^i V\rangle).$$

If n is odd, then all subgroups $\langle R^i V\rangle$ are conjugate, and hence $g(X/\langle R^i V\rangle) = g(X/\langle V\rangle)$ for $i = 1, 2, \ldots, n-1$.

If n is even, then $\langle R^i V\rangle$ and $\langle R^j V\rangle$ are conjugate if and only $i \equiv j \mod 2$. Thus we have $\sum_{i=0}^{n-1} g(X/\langle R^i V\rangle)$ is equal to $ng(X/\langle V\rangle)$ for odd n, and $\frac{n}{2}(g(X/\langle V\rangle) + g(X/\langle R\,V\rangle))$ for even n. Therefore we obtain the desired result.

Let $F(p)$ be a finite field of characteristic p. Denote by $f_{(a,b)}$, $a, b \in F(p)$, $a \neq 0$, the map $x \to ax + b$, $x \in F(p)$. Put $K = \{f_{(a,b)} : a, b \in F(p), a \neq 0\}$ and $N = \{f_{(1,b)} : b \in F(p)\}$. Note that the affine group K is a group admitting partition $\{N, K_{x_0}, x_0 \in F(p)\}$, where K_{x_0} is the stabilizer subgroup of x_0 consisting of transformations $x \to a(x - x_0) + x_0$, $a \neq 0$. Then we have the following result by the method similar to the proof of Application 1.

Application 2 *Let X be a finite graph and the group K acts purely harmonically on X. Then we have*

$$g(X) + (p-1)g(X/K) = g(X/N) + (p-1)g(X/K_0).$$

Following [6] and [8] a graph X is said to be *γ-hyperelliptic* if there is a two fold harmonic map $\varphi : X \to Y$, where graph Y is of genus γ. Each edge of Y has two pre-images under φ and there is an order 2 automorphism J of X, which swaps these pre-images. This automorphism is called *γ-hyperelliptic* involution. Note that γ-hyperelliptic involution acts on X purely harmonically. The case $\gamma = 0$ coincides with the definition of a hyperelliptic graph [2].

Application 3 *Let γ be a nonnegative integer. Let X be a graph of genus g so that $g > 4\gamma + 1$. Suppose J is an automorphism of order two acting purely harmonically on X such that the genus of $X/\langle J\rangle$ is γ. Then these properties define J uniquely and $\langle J\rangle$ is central in the full group of automorphisms of X.*

Proof Suppose J_1 and J_2 are two distinct automorphisms of X with the properties of J. Then J_1 and J_2 generate a dihedral group, $\mathbb{D}_n$ of order $2n$. We set $R = J_1 J_2$ and $V = J_2$. Then, Application 1 gives

$$g + 2g(X/\mathbb{D}_n) = 2\gamma + g(X/\langle R\rangle).$$

The Riemann–Hurwitz formula for graphs applied to $X \to X/\langle R\rangle$ gives

$$g - 1 = n(g(X/\langle R\rangle) - 1) + r,$$

where n is the order of R. But $n \geq 2$ since J_1 and J_2 are distinct and $r \geq 0$. So

$$g - 1 \geq 2(g(X/\langle R\rangle) - 1)$$

or

$$2g(X/\langle R\rangle) \leq g + 1.$$

Since $g(X/\mathbb{D}_n) \geq 0$, we have

$$2g \leq 2g + 4g(X/\mathbb{D}_n) = 4\gamma + 2g(X/\langle R\rangle) \leq 4\gamma + g + 1$$

or

$$g \leq 4\gamma + 1.$$

This contradiction shows that J is unique.

Let T be another automorphism of X. Then, $T^{-1}JT$ has the same properties as J. Thus $J = T^{-1}JT$, and the proof is complete. □

Remark 1 The above statement is a generalization of the hyperelliptic situation, $\gamma = 0$. Recall that by [2] for $g > 1$ the hyperelliptic involution on a graph of genus g is unique.

Remark 2 Application 3 asserts that in the case $g > 4\gamma + 1$ the γ-hyperelliptic involution is also unique.

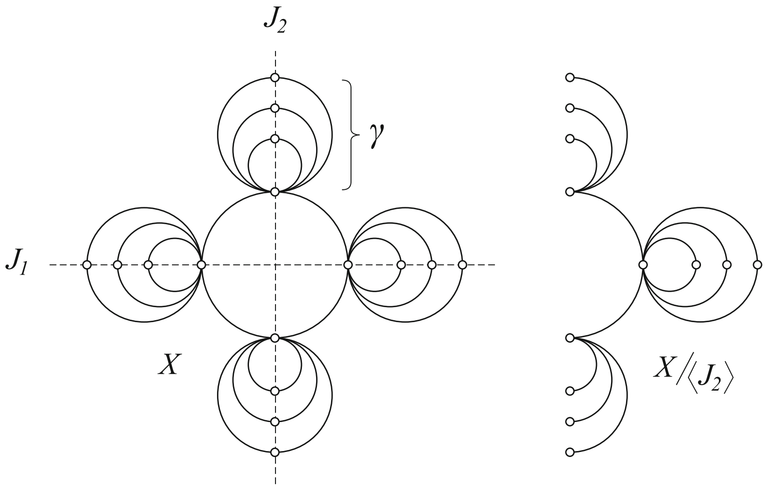

Fig. 1 Graph X and quotient graph $X/\langle J_2\rangle$

The following counterexample shows that if condition $g > 4\gamma + 1$ is not satisfied γ-hyperelliptic involution is not necessary unique. Let X be a graph of genus $4\gamma + 1$, where $\gamma \geq 0$, depicted in the Fig. 1. Graph X has two automorphisms of order 2, J_1 and J_2, acting purely harmonically. The quotient graphs $X/\langle J_i\rangle$, $i = 1, 2$, both have genus γ. If $\gamma = 0$, we get the counterexample to hyperelliptic case. In the latter case $g = 1$.

Application 4 *Let g_i and g_2 be nonnegative integers. Let X be a graph of genus g so that $2g \geq 3g_1 + 3g_2 + 3$. Let X admit two distinct automorphisms A_1 and A_2, acting purely harmonically, both of period two so that the genus of $X/\langle A_i\rangle$ is g_i. Then A_1 and A_2 commute.*

Proof Again, setting $g_0 = g(X/\langle A_1,\ A_2\rangle)$ and $g_3 = g(X/\langle A_1 A_2\rangle)$, by Application 1 we have

$$g + 2g_0 = g_1 + g_2 + g_3.$$

Let the product $A_1 A_2$ have order n. We wish to show that n is two, so suppose $n \geq 3$. The Riemann–Hurwitz formula for $X \to X/\langle A_1 A_2\rangle$ is

$$g - 1 = n(g_3 - 1) + r.$$

Hence, $g - 1 \geq 3(g_3 - 1)$ or $3g_3 \leq g + 2$.

Since $g_0 \geq 0$, we have

$$3g \leq 3g + 6g_0 = 3g_1 + 3g_2 + 3g_3$$

or

$$3g \leq 3g_1 + 3g_2 + g + 2.$$

This contradicts the hypothesis and thus n is two.

The statement of the next application was suggested by W. T. Kiley and adopted by R. D. H. Accola for the Riemann surface automorphisms.

Application 5 *Let γ be a positive integer. Let X be a graph of genus g so that $g > 3\gamma + 2$. Suppose X admits two distinct automorphisms A_1 and A_2, acting purely harmonically, both with period two, so that the genus of $X/\langle A_i \rangle$ is γ, $i = 1, 2$. Then A_1 and A_2 commute. Moreover, $\langle A_1, A_2 \rangle$ and $\langle A_1 A_2 \rangle$ are normal in the full group of automorphisms of X.*

Proof That A_1 and A_2 commute follows immediately from the previous application. To show that $\langle A_1, A_2 \rangle$ and $\langle A_1 A_2 \rangle$ are normal, we show that there is no further A_3 with the properties of A_1 and A_2. Thus conjugation will permute A_1 and A_2 and consequently leave $A_1 A_2$ fixed. Therefore, suppose that A_3 is a third distinct automorphism with the properties of A_1 and A_2. Consider $G_0 = \langle A_1, A_2, A_3 \rangle$. If $A_3 = A_1 A_2$ then Galois isomorphic to $Z_2 \times Z_2$ and Application 1 leads to the contradiction

$$g + 2g(X/G_0) = 3\gamma.$$

Thus Galois isomorphic to $Z_2 \times Z_2 \times Z_2$. Let the seven subgroups of order two be denoted G_i, $i = 1, 2, \ldots, 7$, where $G_i = \langle A_i \rangle$ for $i = 1, 2, 3$. Let g_i be the genus of X/G_i. The statement of Theorem 1 becomes

$$3g + 4g_0 = \sum_{i=1}^{7} g_i.$$

Now $g_1 = g_2 = g_3 = \gamma$. Also, as in Application 4, $2g_i \leq g + 1$ for $i = 4, 5, 6, 7$. Multiplying the above equation by two and putting in this information, we get

$$6g \leq 6g + 8g_0 \leq 6\gamma + 4g + 4.$$

This contradiction completes the proof.

We finish the paper by the following generalisation of Theorem 1 for the groups not necessary admitting a partition. This result was obtained by Accola [10] for Riemann surfaces and was extended by Takeshi Taniguchi [12] to finite group actions on a compact Hausdorff space.

Theorem 2 *Let X be a finite graph of genus g on which G_0 acts purely harmonically and let $G_1, G_2, \dots, G_s$ be subgroups of G_0 such that $G_0 = \bigcup_{k=1}^{s} G_k$. For indices $1 \le i < j < \ldots < k \le s$, put*

$$\begin{aligned} G_{i,\,j,\ldots,k} &= G_i \cap G_j \cap \ldots \cap G_k, \\ g_{i,\,j,\ldots,k} &= g(X/G_{i,\,j,\ldots,k}), \\ n_{i,\,j,\ldots,k} &= |G_{i,\,j,\ldots,k}|. \end{aligned}$$

Put also $g_0 = g(X/G_0),\ n_0 = |G_0|$. Then it holds

$$\begin{aligned} n_0 g_0 = \sum_{1\le i\le s} n_i g_i - \sum_{1\le i<j\le s} n_{i,\,j} g_{i,\,j} + \sum_{1\le i<j<k\le s} n_{i,\,j,\,k} g_{i,\,j,\,k} \\ - \ldots - (-1)^s n_{1,\,2,\ldots,s} g_{1,\,2,\ldots,s}. \end{aligned}$$

Proof For the coverings $X \to X/G_0$ and $X \to X/G_{i,\,j,\ldots,k}$ the Riemann–Hurwitz formula gives

$$\begin{aligned} g - 1 &= n_0(g_0 - 1) + r_0, \\ g - 1 &= n_{i,\,j,\ldots,k}(g_{i,\,j,\ldots,k} - 1) + r_{i,\,j,\ldots,k}, \end{aligned} \tag{6}$$

where

$$\begin{aligned} r_0 &= \sum_{v\in V(X)} (|G_0^v| - 1), \\ r_{i,\,j,\ldots,k} &= \sum_{v\in V(X)} (|G_{i,\,j,\ldots,k}^v| - 1). \end{aligned} \tag{7}$$

Since $G_0 = \bigcup_{k=1}^{s} G_k$, by the inclusion-conclusion principle we obtain

$$\begin{aligned} n_0 = \sum_{1\le i\le s} n_i - \sum_{1\le i<j\le s} n_{i,\,j} \\ + \sum_{1\le i<j<k\le s} n_{i,\,j,\,k} - \cdots - (-1)^s n_{1,\,2,\ldots,s}. \end{aligned} \tag{8}$$

In particular, if group G_0 is the trivial one, we have

$$1 = \sum_{1\le i\le s} 1 - \sum_{1\le i<j\le s} 1 + \sum_{1\le i<j<k\le s} 1 - \cdots - (-1)^s 1. \tag{9}$$

In the same time, since $G_0^v = \bigcup_{k=1}^{s} G_k^v$, we get

$$\begin{aligned} |G_0^v| = \sum_{1\le i\le s} |G_i^v| - \sum_{1\le i<j\le s} |G_{i,\,j}^v| \\ + \sum_{1\le i<j<k\le s} |G_{i,\,j,\,k}^v| - \cdots - (-1)^s |G_{1,\,2,\ldots,s}^v|. \end{aligned} \tag{10}$$

Taking into consideration (9) and (10), and summing through all $v \in V(X)$, from (7) we obtain

$$r_0 = \sum_{1 \le i \le s} r_i - \sum_{1 \le i < j \le s} r_{i,\,j} + \sum_{1 \le i < j < k \le s} r_{i,\,j,\,k} - \cdots - (-1)^s r_{1,\,2,\ldots,s}. \quad (11)$$

Now substitute equations (6) into (11).

$$\begin{aligned}(g-1) - n_0(g_0 - 1) = & \sum_{1 \le i \le s} [(g-1) - n_i(g_i - 1)] \\ & - \sum_{1 \le i < j \le s} [(g-1) - n_{i,\,j}(g_{i,\,j} - 1)] \\ & + \sum_{1 \le i < j < k \le s} [(g-1) - n_{i,\,j,\,k}(g_{i,\,j,\,k} - 1)] \\ & - \cdots - (-1)^s [(g-1) - n_{1,\,2,\ldots,s}(g_{1,\,2,\ldots,s} - 1)].\end{aligned}$$

Eliminate the terms $g - 1$ from each summand by (9). Taking into consideration formula (8) finishes the proof.

Remark 3 Note that the arguments of the paper can be used to investigate "p-gonal discrete" Riemann surfaces. Some results in this direction are already obtained by the first named author in [13].

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